Pebbling in 2-Paths

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Abstract

Graph pebbling is a network model for transporting discrete resources that are consumed in transit. Deciding whether a given configuration on a particular graph can reach a specified target is NP-complete, even for diameter two graphs, and deciding whether the pebbling number has a prescribed upper bound is Π_2^{P}-complete. Recently we proved that the pebbling number of a split graph can be computed in polynomial time. This paper continues the program of finding other polynomial classes, moving away from the large tree width, small diameter case (such as split graphs) to small tree width, large diameter, beginning an investigation on the important subfamily of chordal graphs called k-trees. In particular, we provide a formula for the pebbling number of any 2-path.

Keywords: pebbling number, k-trees, k-paths, Class 0, complexity

1 Introduction

The fundamental question in graph pebbling is whether a given supply (*configuration*) of discrete pebbles on the vertices of a connected graph can satisfy a

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particular set of demands on the vertices. The operation of pebble movement across an edge is called a *pebbling step*: while two pebbles cross the edge, only one arrives at the opposite end, as the other is consumed. The most studied scenario involves the demand of one pebble on a single *root* vertex r. Satisfying this demand is often referred to as *reaching* or *solving* r, and configurations are consequently called either r-*solvable* or r-*unsolvable*.

The *size* $|C|$ of a configuration $C : V \to \mathbb{N} = \{0, 1, \ldots\}$ is its total number of pebbles $\sum_{v \in V} C(v)$. The *pebbling number* $\pi(G) = \max_{r \in V} \pi(G, r)$, where $\pi(G, r)$ is defined to be the minimum number s so that every configuration of size at least s is r-solvable. Simple sharp lower bounds like $\pi(G) \geq n$ and $\pi(G) \geq 2^{\mathsf{diam}(G)}$ are easily derived. Graphs satisfying $\pi(G) = n$ are called *Class* 0 and are a topic of much interest. Recent chapters in [6] and [7] include variations on the theme such as k-pebbling, fractional pebbling, optimal pebbling, cover pebbling, and pebbling thresholds, as well as applications to combinatorial number theory, combinatorial group theory, and p-adic diophantine equations, and also contain important open problems in the field.

Computing graph the pebbling number is difficult in general. The problem of deciding if a given configuration on a graph can reach a particular vertex was shown in [8] and [10] to be NP-complete, even for diameter two graphs ([4]) or planar graphs ([9]). Interestingly, the problem was shown in [9] to be in P for graphs that are both planar and diameter two, as well as for outerplanar graphs (which include 2-trees). The problem of deciding whether a graph G has pebbling number at most k was shown in [10] to be Π_2^{P}-complete.

In contrast, the pebbling number is known for many graphs. For example, in [11] the pebbling number of a diameter 2 graph G was determined to be n or $n+1$. Moreover, [3] and [2] characterized those graphs having $\pi(G) = n+1$, and it was shown in [5] that one can recognize such graphs in quartic time. Beginning a program to study for which graphs their pebbling number can be computed in polynomial time, the authors of [1] produced a formula for the family of split graphs that involves several cases. For a given graph, finding to which case it belongs takes $O(n^{1.41})$ time. The authors also conjectured that the pebbling number of a chordal graph of bounded diameter can be computed in polynomial time.

In contrast to the small diameter, large tree width case of split graphs, we turn here to chordal graphs with large diameter and small tree width. In this paper we study 2-paths, the sub-class of 2-trees whose graphs have exactly two simplicial vertices, and prove an exact formula that can be computed in linear time.

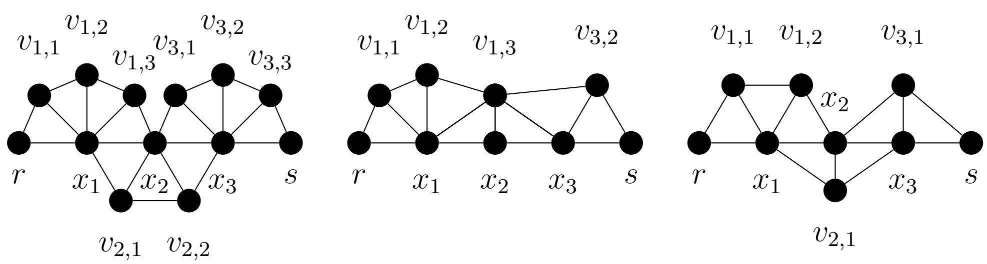

Fig. 1. An overlapping fan graph (left) of diameter 4; fans F_1 and F_3 are same-sided (upper) fans, while F_2 is a lower fan, opposite-sided from F_1 and F_3. An overlapping fan graph with unpleasant (center: $v_{1,3} = v_{2,1} = v_{3,1}$) and pleasant (right: relabeled) shortest rs-paths.

2 Preliminary Definitions and Results

A *simplicial* vertex in a graph is a vertex whose neighbors form a complete graph. It is *k-simplicial* if it also has degree k. A *k-tree* is a graph G that is either a complete graph of size k or has a k-simplicial vertex v for which $G - v$ is a k-tree. A *k-path* is a k-tree with exactly two simplicial vertices. For the purpose of our work we derive a new characterization of 2-trees that facilitates the analysis of its pebbling number.

Let $P = x_0, x_1, \ldots, x_{d-1}, x_d$ be a shortest rs-path between two vertices $r = x_0$ and $s = x_d$ of G, where $d = \mathsf{dist}(r, s) = \mathsf{diam}(G)$. A *fan* is a subgraph F of G consisting of a path $Q = a, v_1, \ldots, v_k, c$ and another vertex b that is adjacent to every vertex of Q, with the extra condition that a, b, c is a subpath of P; we also stipulate that a is closer to r than is c. We call F an *ac-fan*, with *fan vertices* $F' = \{v_1, \ldots, v_k\}$. The graph G is an *overlapping fan graph* if, for every $0 < i < d$, there is an $x_{i-1}x_{i+1}$-fan F_i in G, and every vertex of G is in some fan of G. In addition, we must have $|F'_i \cap F'_{i+1}| \in \{0, 1\}$.

In the 0 (resp. 1) case, we say that F_i and F_{i+1} are *opposite-sided* (resp. *same-sided*) fans. We always call F_1 an *upper* fan, which determines all further fans as upper or *lower* (opposite-sided from upper) — see Figure 1. In the $|F'_i \cap F'_{i+1}| = 1$ case, if $F'_j = \{v_{j,1}, \ldots, v_{j,k_j}\}$ then $v_{i,k_i} = v_{i+1,1}$. Finally, $|F'_{i-1} \cap F'_{i+1}| \leq 1$, with equality if and only if $k_i = 1$, while $|F'_i \cap F'_{i+3}| = 0$. Note that we can choose P so that $|F'_{i-1} \cap F'_{i+1}| = 0$ by swapping the names of vertices x_i and $v_{i,1}$, changing the fans F_{i-1}, F_i, and F_{i+1} from being same-sided to F_i being opposite-sided from F_{i-1} and F_{i+1}. Such a choice of path P is called *pleasant* (see Figure 1).

The following lemma is straightforward to prove by induction.

Lemma 2.1 *A graph G is a 2-path if and only if it is an overlapping fan graph.*

With respect to pebbling configurations, we define an *empty vertex* (or *zero*) to be a vertex with no pebbles on it. A *big vertex* has at least two pebbles on it; of course, in an r-unsolvable configuration, every path from a big vertex to the root r must contain at least one zero. A *huge vertex* v has at least $2^{\mathsf{dist}(v,r)}$ pebbles on it; of course, no r-unsolvable configuration has a huge vertex. The *cost* of a pebbling solution is the number of pebbles lost during the pebbling steps of that solution, plus one for the pebble that reaches r. A *cheap solution* is a solution of cost at most $2^{\mathsf{diam}(G)}$.

The *t-pebbling number* $\pi_t(G)$ is the minumum number s so that every configuration of size s is t-fold solvable (i.e., can place t pebbles on any root). The t-pebbling number is related to the fractional pebbling number, which measures the limiting average cost of repeated solutions; i.e. $\lim_{t\to\infty} \pi_t(G)/t$. It is also used as a powerful inductive tool for computing the pebbling number. The following theorem was proven in [5].

Theorem 2.2 *[5] If G is a graph of diameter 2 then $\pi_t(G) \le \pi(G) + 4t - 4$.*

In what follows we outline the key lemmas and ideas of our proof of the pebbling number for 2-paths.

3 Result

We first note that diameter 2 2-paths are Class 0.

Lemma 3.1 *If G is a 2-path of diameter 2 on n vertices then $\pi(G) = n$. In particular, every size n configuration has a cheap solution to any root.*

This is not difficult to argue. Without huge vertices there must be several big vertices, which are close enough to any root to reach it independently or cooperatively.

Next we prove the Cheap Lemma, which is the central engine of our main theorem. The Cheap Lemma shows that large configurations have very nice solutions, so that we can use it repeatedly to achieve our result. Given a vertex r of a graph G on n vertices, define the function $p_t(G,r) = t2^d + n - 2d$, where $d = \mathsf{diam}(G)$.

Lemma 3.2 **[Cheap Lemma]** *Suppose that G is a 2-path on n vertices with simplicial vertex r, and let C be a configuration on G of size at least $p_t(G,r)$. Then*

(i) *C is r-solvable and*

(ii) *if $t > 1$ there is a cheap r-solution.*

Proof sketch. When $d = 2$, the result is taken care of by Lemma 3.1. So we will assume that $d > 2$ and use induction. Suppose that $|C| = p_t(G, r)$ and let $P = r, x_1, \ldots, x_{d-1}, s$ be a pleasant shortest rs-path between the two simplicial vertices of G. Label G by its overlapping fan graph labeling, so that $V(F_i) = \{x_{i-1}, x_i, x_{i+1}, v_{i,1}, \ldots, v_{i,k_i}\}$ and Q_i is the path $x_{i-1}, v_{i,1}, \ldots, v_{i,k_i}, x_{i+1}$. Let G' be the restriction of G to the n' vertices of $\cup_{i \geq 2} V(F_i)$, with C' denoting the restriction of C to G'. We further use the abbreviations $C_1 = C(F_1)$ and $n_1 = n(F_1)$. Notice that $\mathsf{diam}(G') = d - 1$, so that the Cheap Lemma holds for G'.

If $C(x_1) \geq 1$, $C(x_2) \geq 2$, or $C(v_{1,j}) \geq 2$ for some j, then we can place a pebble on x_1. Let's suppose that F_1 and F_2 are opposite-sided. If $|C'| - (1, 2, 0) \geq p_{2t-1}(G', x_1)$, where the coordinates correspond, in order, to the three cases above, then we can place another pebble on x_1, cheaply if $2t-1 \geq 2$. Thus we can solve r, and at a cost of at most $2^{d-1} + 2 \leq 2^d$ when $t \geq 2$. Otherwise, $|C'| - (1, 2, 0) \leq p_{2t-1}(G', x_1) - 1$. That is, $|C'| \leq [(2t-1)2^{d-1} + n' - 2(d-1)] + (0, 1, -1)$. Thus $|C_1| \geq |C| - |C'| + (1, 2, 0) \geq 2^{d-1} + (n_1 - 2) - 2 + (1, 1, 1) = n_1 + (2^{d-1} - 3) > n_1$, which means by Lemma 3.1 that we can solve r with cost at most $4 < 2^d$. The case for same-sided F_1 and F_2 is similar.

On the other hand, if $C(x_1) = 0$, $C(x_2) \leq 1$, and $C(v_{1,j}) \leq 1$ for all j, then $C(r, v_{1,1}, \ldots, v_{1,k_1-1}) \leq k_1 - 1$. Define $\sigma = 1$ (0) if F_2 is same-sided (opposite-sided) as F_1, so that $|C'| \geq p_3(G', x_1)$ when $t \geq 2$. Thus we can use induction to place a pebble cheaply on x_1, after which at least $p_3(G', x_1) - 2^{d-1} = p_2(G', x_1)$ pebbles remain, and so we can place another pebble cheaply on x_1, and hence one cheaply on r.

The analysis proceeds similarly for the case in which $t = 1$, taking into account the number o' of zeros in $\{v_{1,1}, \ldots, v_{1,k_1}\}$. When $o' - 1 - \sigma \geq 0$ one can use induction twice to place two pebbles on x_1, and hence one on r. Thus we may assume that $o' - 1 - \sigma < 0$. If $o' = 0$ then one can solve r by placing two pebbles on x_2 and then moving to r along Q_1. Otherwise $o' = 1$, which means that $\sigma = 1$. We use induction twice again, as follows. When v_{1,k_1} is not empty, we solve r by placing two pebbles on x_1. When v_{1,k_1} is empty, we place two pebbles on v_{1,k_1} and solve r via Q_1. $\square$

Theorem 3.3 *If G is a 2-path of diameter d on n vertices then $\pi_t(G) = t2^d + n - 2d$.*

Proof sketch. It is not difficult to argue that $\pi_t(G,r)$ is maximized when r is simplicial; then the upper bound follows from the Cheap Lemma 3.2 by induction. The lower bound comes from the existence of a t-fold r-unsolvable configuration of size $t2^d+n-2d-1$. Let $\mathcal{S}$ be a collection of pairwise disjoint separating edges, exactly one per fan. Define C by placing $t2^d-1$ pebbles on one simplicial vertex and one pebble on each of the remaining vertices, except none on r or $V(\mathcal{S})$. It can be shown by induction that t pebbles cannot be moved to r. □

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4 Appendix

Here we give the full proof of the Cheap Lemma.

Proof. When $d = 2$, the result is taken care of by Lemma 3.1. So we will assume that $d > 2$ and use induction. Suppose that $|C| = p_t(G, r)$ and let $P = r, x_1, \ldots, x_{d-1}, s$ be a pleasant shortest rs-path between the two simplicial vertices of G. Label G by its fan graph labeling, so that $V(F_i) = \{x_{i-1}, x_i, x_{i+1}, v_{i,1}, \ldots, v_{i,k_i}\}$ and Q_i is the path $x_{i-1}, v_{i,1}, \ldots, v_{i,k_i}, x_{i+1}$. Let G' be the restriction of G to the n' vertices of $\cup_{i\geq 2} V(F_i)$, with C' denoting the restriction of C to G'. We further use the abbreviations $C_1 = C(F_1)$ and $n_1 = n(F_1)$. Notice that $\mathsf{diam}(G') = d - 1$, so that the Theorem holds for G'.

If $C(x_1) \geq 1$, $C(x_2) \geq 2$, or $C(v_{1,j}) \geq 2$ for some j, then we can place a pebble on x_1. Let's suppose that F_1 and F_2 are opposite-sided. If $|C'| - (1, 2, 0) \geq p_{2t-1}(G', x_1)$, where the coordinates correspond, in order, to the three cases above, then we can place another pebble on x_1, cheaply if $2t-1 \geq 2$. Thus we can solve r, and at a cost of at most $2^{d-1} + 2 \leq 2^d$ when $t \geq 2$. Otherwise, $|C'| - (1, 2, 0) \leq p_{2t-1}(G', x_1) - 1$. That is, $|C'| \leq [(2t - 1)2^{d-1} + n' - 2(d - 1)] + (0, 1, -1)$. Thus $|C_1| \geq |C| - |C'| + (1, 2, 0) \geq 2^{d-1} + (n_1 - 2) - 2 + (1, 1, 1) = n_1 + (2^{d-1} - 3) > n_1$, which means by Lemma 3.1 that we can solve r with cost at most $4 < 2^d$. If instead F_1 and F_2 are same-sided, then all these calculations are the same, except in the instance of the third case above for which $j = k_1$. In this instance we consider when $|C'| - 2 \leq p_{2t-1}(G', x_1) - 1$; i.e., $|C'| \leq [(2t - 1)2^{d-1} + n' - 2(d - 1)] + 1$. Thus $|C_1| \geq |C| - |C'| + 2 \geq 2^{d-1} + (n_1 - 3) - 2 + 1 = n_1 + (2^{d-1} - 4) \geq n_1$, which means by Lemma 3.1 that we can solve r with cost at most $4 < 2^d$.

On the other hand, if $C(x_1) = 0$, $C(x_2) \leq 1$, and $C(v_{1,j}) \leq 1$ for all j, then $C(r, v_{1,1}, \ldots, v_{1,k_1-1}) \leq k_1 - 1$. Define $\sigma = 1$ (0) if F_2 is same-sided (opposite-sided) as F_1, so that

$$\begin{aligned}
|C'| &\geq |C| - (k_1 - \sigma) \\
&= (t2^d + n - 2d) - (k_1 - \sigma) \\
&= (2t)2^{d-1} + n' - 2(d - 1) - 1 \\
&\geq (3)2^{d-1} + n' - 2(d - 1) \\
&= p_3(G', x_1) \ ,
\end{aligned}$$

when $t \geq 2$. Thus we can use induction to place a pebble cheaply on x_1, after which at least $p_3(G', x_1) - 2^{d-1} = p_2(G', x_1)$ pebbles remain, and so we can place another pebble cheaply on x_1, and hence one cheaply on r.

It remains to handle the case in which $t = 1$. Here we define o to be the number of zeros in $\{v_{1,1}, \ldots, v_{1,k_1}, x_2\}$, so that $|C_1| = n_1 - 2 - o$, and set o' to be the number of those zeros other than x_2 (i.e. $o - o' = 1 - C(x_2)$). Now we have

$$\begin{aligned}
|C'| &\geq |C| - |C_1| + C(x_2) \\
&= (2^d + n - 2d) - (n_1 - 2 - o) + C(x_2) \\
&= (2)2^{d-1} + (n' - 2 - \sigma) - 2d + 2 + o + C(x_2) \\
&= [(2)2^{d-1} + n' - 2(d-1)] + [C(x_2) + o - 2 - \sigma] \\
&= p_2(G', x_1) + [o' - 1 - \sigma].
\end{aligned}$$

If $o' - 1 - \sigma \geq 0$ then $|C'| \geq p_2(G', x_1)$, which means, by using induction twice, that we can place two pebbles on x_1. Therefore, we can solve r.

Otherwise, we have $o' - 1 - \sigma < 0$, which means that $o' \leq \sigma$. If $o' = 0$, that is $C(v_{1,j}) = 1$ for all j, then we will show that it is possible to place two pebbles on x_2, from which we solve r by moving pebbles from x_2 along Q_1. Indeed, this is so if $C(x_2) = 1$ and Q_2 has a big vertex, or if Q_2 contains either a vertex with four pebbles or two big vertices, so we assume otherwise. In this case, we have $C((F_1 \cup F_2) - G'') \leq |(F_1 \cup F_2) - G''|$, where G'' is the restriction of G to the n'' vertices of $\cup_{i \geq 3} V(F_i)$. For the restriction C'' of C to G'', this implies that

$$\begin{aligned}
|C''| &= |C| - C((F_1 \cup F_2) - G'') \\
&\geq 2^d + n'' - 2d \\
&= [2^{d-1} + n'' + 2(d-2)] + [2^{d-1} - 4] \\
&\geq p_2(G'', x_2) \ ,
\end{aligned}$$

since $d \geq 3$, and so we can place two pebbles on x_2.

We are left now with the final case in which $o' = 1$ (exactly one $v_{1,j}$ is empty), which means that $\sigma = 1$ (F_1 and F_2 are same-sided, so that $v_{1,k_1} = v_{2,1}$).

If v_{1,k_1} is not empty then $k_1 \geq 2$, and so

$$\begin{aligned}
|C'| &= |C| - (k_1 - 2) \\
&= (2)2^{d-1} + (n - k_1) - 2(d-1) \\
&= p_2(G', x_1) \ .
\end{aligned}$$

This means, by using induction twice, that we can place two pebbles on x_1, and hence one on r.

If instead v_{1,k_1} is empty then set $\hat{G} = G' - x_1$ and $\hat{C} = C(\hat{G})$, so that

$$\begin{aligned} |\hat{C}| &= |C| - (k_1 - 1) \\ &= (2)2^{d-1} + (n - k_1 - 1) - 2(d-1) \\ &= p_2(\hat{G}, v_{1,k_1}) \ . \end{aligned}$$

This means, by using induction twice, that we can place two pebbles on v_{1,k_1}, and hence one on r (via Q_1).

This completes the proof. □

4.1 Extra References

The following references were omitted in the body of the paper due to space limitation. We list them here for the convenience of the referees.

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