GRAPH PEBBLING

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ABSTRACT. The graph pebbling model we study here was born as a method for solving a combinatorial number theory conjecture of Erdős and Lemke and has since been applied to problems in combinatorial group theory and p-adic diophantine equations. Related pebbling models have found applications in computational complexity, compiler theory, graph searching, sparse matrix factorization, and computational geometry. The subject has grown in the last two decades into a network optimization model for the transportation of consumable resources. When two pebbles are moved across an edge, only one of them arrives at the other end while the other is lost, as if to a toll. The most basic question asks if it is possible to move from one configuration of pebbles to another via this pebbling rule. Here we present the current state of knowledge regarding best, worst, and average case scenarios of this paradigm, as well as random, fractional, and other versions. Along the way we survey the important methods and algorithmic considerations.

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Date: March 25, 2015.
1991 *Mathematics Subject Classification.* 90C27 (05C57, 05C85, 11B75).

Introduction

Graph Pebbling is a network optimization model for the transportation of resources that are consumed in transit. Electricity, heat, or other energy may dissipate as it moves from one location to another, oil tankers may use up some of the oil it transports, information may be lost as it travels through its medium, or military troops may be lost while moving through a region. The central problem in this model asks whether discrete pebbles from one set of vertices can be moved to another while pebbles are lost in the process.

A typical question asks how many pebbles are necessary to guarantee that, from any configuration of that many pebbles, one can move a pebble to ("solve") any particular vertex. A fractional version asks for the limiting behavior of the average number of pebbles used per "solution". Instead of placing the initial pebbles cleverly, if the original configuration is chosen at random then we can wonder what the probability is that every vertex can be solved, which gives rise to the notion of a threshold, which divides almost sure success and almost sure failure. One may also ask: how few pebbles can be used so that, from some configuration of that many pebbles, one can move a pebble to any particular vertex; how many pebbles are required to guarantee that, from any configuration of that many pebbles, some pebble can travel at least some fixed distance; and other questions. Good surveys of the subject can be found in [51, 53, 55].

Various rules for pebbling steps have been studied for years and have found applications in a wide array of areas. One version, dubbed *black and white* pebbling, was applied to computational complexity theory in studying time-space tradeoffs (see [48, 69]), as well as to optimal register allocation for compilers (see [72]). Connections have been made also to pursuit and evasion games and graph searching (see [58, 68]). Another (*black* pebbling) is used to reorder large sparse matrices to minimize in-core storage during an out-of-core Cholesky factorization scheme (see [36, 61, 63]). A third version yields results in computational geometry in the rigidity of graphs, matroids, and other structures (see [39, 73]). The rule we study here originally produced results in combinatorial number theory and combinatorial group theory (the existence of zero sum subsequences — see [15, 28]) and have recently been applied to finding solutions in p-adic diophantine equations (see [62]). Most of these rules give rise to computationally difficult problems.

Notation. All graphs considered are simple and connected. We follow fairly standard graph terminology (e.g. [76]), with a graph $G = (V, E)$ having $n = n(G)$ vertices $V = V(G)$, with edges $E = E(G)$. The *eccentricity* $\mathsf{ecc}(G, r)$ for a vertex $r \in V$ equals $\max_{v \in V} \mathsf{dist}(v, r)$, where $\mathsf{dist}(x, y)$ denotes the length (number of edges) of the shortest path from x to y; the *diameter* $\mathsf{diam}(G) = \max_{r \in V} \mathsf{ecc}(G, r)$. Other well known graph parameters such as girth, connectivity, radius, and domination number of a graph G are written $\mathsf{gir}(G)$, $\kappa(G)$, $\mathsf{rad}(G)$, and $\mathsf{dom}(G)$, respectively, and the minimum degree of G is denoted $\delta(G)$. When H is a subgraph of G, we write $G - H$ to denote the graph having vertices $V(G - H) = V(G)$ and edges $E(G - H) = E(G) - E(H)$.

Common graphs (on n vertices) such as the complete graph, path, and cycle are denoted K_n, P_n, and C_n, respectively. The d-dimensional cube Q^d has $n = 2^d$ vertices. Finally, we write lg for the base 2 logarithm, and $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$ for the sets of nonnegative integers and reals.

FIGURE 1. Two r-unsolvable configurations on the path P_7.

1. SOLVABILITY

Here we develop the notion of moving from one configuration of pebbles to another via pebbling steps.

1.1. **Basic Definitions.** A *configuration* C on a graph G is a function $C : V(G) \rightarrow \mathbb{N}$. The value $C(v)$ signifies the number of pebbles at vertex v. A vertex with no pebbles on it is called *empty* and a vertex with more than one pebble on it is called *big*. The *size* $|C|$ of a configuration C on a graph G is the total number of pebbles on G; i.e. $|C| = \sum_{v \in V(G)} C(v)$. We also write $C(S) = \sum_{v \in S} C(v)$ for a subset $S \subseteq V(G)$ of vertices. For an edge $\{u, v\} \in E(G)$, if u has at least two pebbles on it, then a *pebbling step from* u *to* v removes two pebbles from u and places one pebble on v. That is, if C is the original configuration, then the resulting configuration C' has $C'(u) = C(u) - 2$, $C'(v) = C(v) + 1$, and $C'(x) = C(x)$ for all $x \in V(G) - \{u, v\}$. A pebbling step from u to v is r*-greedy* if $\mathsf{dist}(v, r) < \mathsf{dist}(u, r)$. It is r*-semigreedy* if $\mathsf{dist}(v, r) \leq \mathsf{dist}(u, r)$.

We say that a configuration C on G is r*-solvable* if it is possible from C to place a pebble on r via pebbling steps. It is r*-unsolvable* otherwise. More generally, for a configuration D, we say that C is D*-solvable* if it is possible to perform pebbling steps from C to arrive at another configuration C' for which $C'(v) \geq D(v)$ for all $v \in V(G)$. It is D*-unsolvable* otherwise. We denote by $\vec{G}(\sigma)$ the directed subgraph of G induced by a set σ of pebbling steps. We say that a configuration C on G is k*-fold* r*-solvable* if it is possible from C to place k pebbles on r via pebbling steps. Note that the k-fold r-solvability of C is the specific instance of D-solvability for which D has k pebbles on r and none elsewhere, a configuration we denote by $k\mathbf{I}_r$ (or $\mathbf{I}_r$ when $k = 1$).

For a graph G and a particular *root* vertex r, the *rooted pebbling number* $\pi(G, r)$ is defined to be the minimum number t so that every configuration C on G of size t is r-solvable. The rooted k-pebbling number $\pi_k(G, r)$ is defined analogously for k-fold solvability. Note that the configuration that places one pebble on every vertex except r shows that $\pi(G, r) \geq n(G)$ always. This is often referred to as the **Vertex Bound**. An example of this bound being tight is the complete graph; the Pigeonhole Principle implies that $\pi(K_n, r) = n$ for all r.

Because the r-solvability of a configuration is not destroyed by adding edges, we have for every root vertex r that $\pi(H, r) \geq \pi(G, r)$ whenever H is a connected, spanning subgraph of G. We call this the **Subgraph Bound**.

Figure 1 shows two configurations on the path P_7, neither of which are r-solvable. Section 1.2 will develop methods to explain why in many cases, but the reader can probably construct a nice argument here. The configuration on the right shows that $\pi(P_7, r) \geq 19$; in fact, it is not difficult to argue that $\pi(P_7, r) = 19$, and that the largest rooted pebbling number $\pi(P_7, r') = 64$ occurs when r' is one of the endpoints of the path. In general, the configuration that places $2^{\mathsf{ecc}(G,r)} - 1$ pebbles on a vertex of maximum distance from r witnesses the **Distance Bound** $\pi(G, r) \geq 2^{\mathsf{ecc}(G,r)}$.

A sequence of paths $\mathcal{P} = (P[1], \ldots, P[h])$ is a *maximum* r*-path partition* of a rooted tree (T, r) if $\mathcal{P}$ forms a partition of $E(T)$, r is a leaf of $P[1]$, $T_i = \cup_{j=1}^{i} P[j]$ is a tree for all $1 \leq i \leq h$, and $P[i]$ is a maximum length path in $T - T_{i-1}$, among all

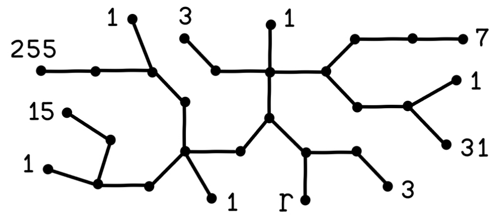

FIGURE 2. An r-unsolvable configuration on a tree.

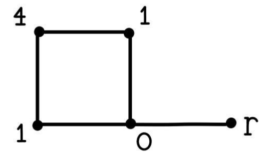

FIGURE 3. An r-solvable configuration with no tree solution.

such paths with one endpoint in T_{i-1}, for all $1 \le i \le h$. We define the function $f(T,r) = \sum_{i=1}^{h} 2^{l_i} - h + 1$, where $(l_1, \ldots, l_h)$ is the sequence of lengths $l_i = \mathsf{diam}(P[i])$ in a maximum r-path partition $\mathcal{P}$ of a rooted tree (T,r). Figure 2 shows an r-unsolvable configuration of size $f(T,r)-1$ that is based on a maximum r-path partition: for each path $P[i]$, $2^{l_i}-1$ pebbles are placed on the endpoint not in T_{i-1}. For example $P[1]$ is the path from the leaf with 255 pebbles on it to r, $P[2]$ is the potion of the path from the leaf with 31 pebbles on it to r that shares no edges with $T_1 = P[1]$, etc.

Theorem 1.1. [15] *The rooted pebbling number of a tree T is $\pi(T,r) = f(T,r)$.*

We leave the proof to the exercises. The key is to define $f_k(T,r) = f(T,r) + (k-1)2^{l_1}$ and prove more generally that $\pi_k(T,r) = f_k(T,r)$. At least two induction proofs are possible: one that removes the root (or, equivalently, includes a copy of r in each component of $G-r$) and one that removes a minimum length path in the maximum r-path partition.

Because the exact pebbling number for trees is known, a rooted spanning of a graph makes an excellent choice for use with the Subgraph Bound. However, it is not sufficient to consider trees only. That is, for a given set σ of pebbling steps, let $G(\sigma)$ be the set of edges of G that are used by σ, and let $\vec{G}(\sigma)$ be the multiset of directed versions of the edges of $G(\sigma)$ (some edges are traversed more than once), with orientations given by the directions of the pebbling steps. We say that a configuration C is *tree-solvable* if $G(\sigma)$ is a tree for some solution σ. Then Figure 3 shows that not every configuration is tree-solvable. However, something tree-like is true.

Lemma 1.2. [No-Cycle Lemma] [13] *If a configuration C is D-solvable then there exists a D-solution σ for which $\vec{G}(\sigma)$ is acyclic.*

Furthermore, recall that a pebbling step from u to v is r- *(semi-) greedy* if $\mathsf{dist}(v,r) < \mathsf{dist}(u,r)$ (resp. $\le$). A set of pebbling steps is r- *(semi-) greedy* if every one of its steps is r- (semi-) greedy, a configuration is r- *(semi-) greedy* it is has an r- (semi-) greedy solution, and a graph G is r- *(semi-) greedy* if every configuration of size at least $\pi(G,r)$ is r- (semi-) greedy. We also say that G is *(semi-) greedy* if it is r- (semi-) greedy for every choice of r. The No-Cycle Lemma shows that every tree is greedy. While it is

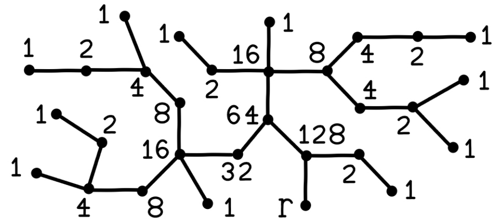

FIGURE 4. A rooted tree with a nonbasic r-strategy.

not difficult to construct configurations on large cycles that are not r-semi-greedy, even cycles are greedy and odd cycles are semi-greedy (see the exercises).

The No-Cycle Lemma gives structure to minimum-sized r-solutions. The following lemma gives structure to maximum-sized r-unsolvable configurations. A *thread* in a graph G is a subpath of G whose vertices have degree two in G. A thread with all its pebbbles sitting on one or two adjacent of its vertices is called *squished*.

Lemma 1.3. [Squishing Lemma] [13] *For every root vertex r of a graph G there is a maximum-sized r-unsolvable configuration such that each thread not containing r is squished.*

The proof uses the notion of squishing moves. A *squishing move* removes 1 pebble from each of two vertices on a thread and places 2 pebbles on some vertex between them on the thread. By using the No-Cycle Lemma, one can show that if C is r-unsolvable and C' is obtained from C by performing a squishing move, then C' is also r-unsolvable.

The Squishing Lemma can be used to give a relatively short proof of the following theorem.

Theorem 1.4. [67] *For every root vertex r in the cycle C_n, we have $\pi(C_{2k}, r) = 2^k$ for all $k \geq 2$ and $\pi(C_{2k+1}, r) = \lceil (2^{k+2} - 1)/3 \rceil$ for all $k \geq 1$.*

This follows because, for an even cycle, a squished r-unsolvable configuration has all its pebbles on one "half" of the cycle, which is the path P_k. For an odd cycle, if this is not the case then the pebbles sit on the two vertices opposite r, in which case moving half the pebbles from the smaller to the larger vertex suffices.

1.2. **Weight Functions.** Weight functions can be used to provide both upper and lower bounds on rooted pebbling numbers of graphs. Let wt denote the *weight function* $wt : V(G) \to \mathbb{R}_{\geq 0}$ defined by $wt(v) = 2^{-\mathsf{dist}(v,r)}$ for all $v \in V(G)$.

More generally, for a tree T rooted at a vertex r we define the *parent* of vertex $v \in V(T) - \{r\}$ to be the unique neighbor v^+ of v for which $\mathsf{dist}(v^+, r) = \mathsf{dist}(v, r) - 1$. In this case v is a *child* of v^+. We say that a rooted subtree (T, r) of (G, r) is an *r-strategy* if associated with it is a weight function $\mathsf{w} : V(G) \to \mathbb{R}_{\geq 0}$ having the properties that $\mathsf{w}(v) = 0$ for all $v \notin V(T)$ and $\mathsf{w}(v^+) \geq 2\mathsf{w}(v)$ for every vertex $v \neq r$. The r-strategy T is *basic* if equality holds for all such $v \in V(T)$ (for example, the strategy in Figure 4 is nonbasic). For a rooted graph (G, r) with r-strategy (T, w), we say that the *weight of a vertex* v is $\mathsf{w}(v)$ when $v \in T$ and 0 otherwise, and define the *weight of a configuration* C on G to be

$$\mathsf{w}(C) = \sum_{v \in V(G)} C(v)\mathsf{w}(v).$$

Note that wt is the weight function for any breadth-first search spanning tree of (G, r), where $wt(r) = 1$. The following proposition is straighforward.

Proposition 1.5. *If C is a configuration on the rooted graph (G, r) and C' is the configuration obtained from C after a pebbling step from u to v then, for any basic r-strategy (T, w) of (G, r) containing the edge $\{u, v\}$, we have $\mathsf{w}(C') \leq \mathsf{w}(C)$, with equality if and only if $\mathsf{w}(v) = 2\mathsf{w}(u)$ (when $\mathsf{w} = wt$ this means that the step is greedy).*

Since a configuration with a pebble on r has weight 1, it follows (by using $\mathsf{w} = wt$ on a breadth-first search spanning tree T) that other r-solvable configurations can only have greater weight.

Corollary 1.6. *If C is an r-solvable configuration on G then $wt(C) \geq 1$.*

The typical use of Corollary 1.6 is in contrapositive form: a configuration C with weight less than 1 is r-unsolvable, and hence $\pi(G, r) > |C|$. It turns out that weight characterizes solvability on paths. We leave the proof to the exercises.

Theorem 1.7. *A configuration C on a path rooted at a leaf r is r-solvable if and only if $wt(C) \geq 1$.*

The same theorem does not hold for trees. Indeed, consider the graph $Y = K_{1,3}$ with root r at one leaf and unknown configuration C with 4 pebbles placed only among the other two leaves. Then we know that $wt(C) = 1$, but C is r-solvable if and only if it has no *odd* vertices; i.e. vertices with an odd number of pebbles. Moreover, if one views the r-unsolvable configurations on Y as points in $\mathbb{N}^3$ (we ignore a coordinate for r since $C(r) = 0$ always), then the r-solvable configuration with 2 pebbles on each leaf is a convex combination of the two r-unsolvable configurations with 1 and 3 pebbles on them instead. More on this idea in a moment.

We denote by $\mathbf{J}_r$ the configuration on any rooted graph (G, r) having no pebbles on r and one pebble on every other vertex. The main use of strategies is to provide upper bounds on rooted pebbling numbers. The following lemma is crucial to this, and can be proved easily by induction (see the exercises).

Lemma 1.8. [Weight Function Lemma] [52] *Let (T, w) be an r-strategy of the rooted graph (G, r) and suppose that C is an r-unsolvable configuration on G. Then $\mathsf{w}(C) \leq \mathsf{w}(\mathbf{J}_r)$.*

For a rooted graph (G, r) on n vertices, let $\mathcal{C}$ be the set of all r-unsolvable configurations on G, viewed as points in $\mathbb{N}^{n-1}$: each $C \in \mathcal{C}$ is identified with the coordinates $(C(v_2), \ldots, C(v_n))$, where $V(G) = \{r, v_2, \ldots, v_n\}$. The convex hull of $\mathcal{C}$ in $\mathbb{R}^{n-1}$ is called the *r-unsolvability polytope* of G, denoted $\mathbf{U}(G, r)$. The configurations on the graph Y above show that, sometimes, $\mathbf{U}(G, r)$ also contains r-solvable configurations. However, none of its extreme points are r-solvable, which gives rise to some linear optimization ideas. First note that $\pi(G, r) = 1 + \max\{|C| \mid C \in \mathbf{U}(G, r)\}$.

Define the *r-strategy polytope* $\mathbf{T}(G, r)$ in $\mathbb{R}^{n-1}$ by the set of linear inequalities given by the Weight Function Lemma over all r-strategies. For a polytope $\mathbf{P}$ of configurations on a rooted graph (G, r) define $z_{\mathbf{P}}(G, r) = \max_{C \in \mathbf{P}} |C|$ and $\pi_{\mathbf{P}}(G, r) = \lfloor z_{\mathbf{P}}(G, r) \rfloor + 1$. The Weight Function Lemma implies that $\mathbf{U}(G, r) \subseteq \mathbf{T}(G, r)$, which proves the following.

Proposition 1.9. [52] *For every rooted graph (G, r) we have $\pi(G, r) = \pi_{\mathbf{U}}(G, r) \leq \pi_{\mathbf{T}}(G, r) \leq \pi_{\mathbf{T}'}(G, r)$, where $\mathbf{T}'$ is any polytope containing $\mathbf{T}$.*

The use of Proposition 1.9 is called the *weight function method*. It is often applied with $\mathbf{T}'$ defined by very few (frequently only $\mathsf{deg}(r)$) r-strategies, as in Figure 5 (proving

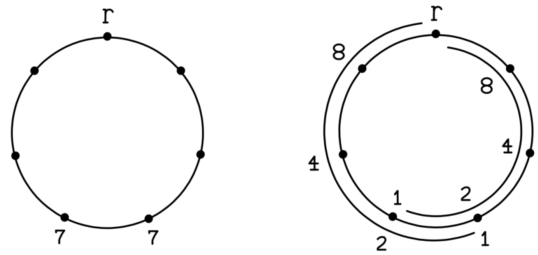

FIGURE 5. A rooted cycle with (left) its maximum r-unsolvable configuration and (right) two basic r-strategies.

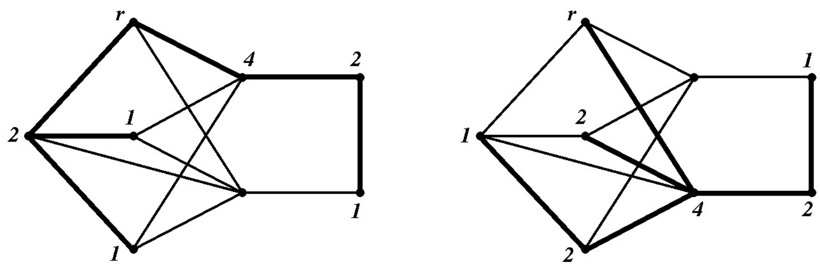

FIGURE 6. Two basic strategies that prove that $\pi(L, r) = 8$.

that $\pi(C_7, r) = 15$). One can use this method to prove that $\pi(L, r) = 8$ for all r (L is called the *Lemke* graph, to be discussed in Subsection 2.1), an example of which is shown in Figure 6. One special consequence of Lemma 1.8 is the following.

Lemma 1.10. [Uniform Covering Lemma] [52] *Let G be a graph on n vertices. If some collection of r-strategies $\{(T_i, \mathsf{w}_i)\}_{i=1}^k$ has the property that there is a constant c such that, for every $v \in V(G) - \{r\}$, we have $\sum_{i=1}^k \mathsf{w}_i(v) = c$, then $\pi(G, r) = n$.*

Proof. The Weight Function Lemma inequality can be rewritten as

$$\sum_{v \in V(G)-\{r\}} C(v)\mathsf{w}(v) \leq \sum_{v \in V(G)-\{r\}} \mathsf{w}(v).$$

Given the r-strategies $\{(T_i, \mathsf{w}_i)\}_{i=1}^k$, we can sum all their inequalities together to obtain

$$\sum_{i=1}^k \sum_{v \in V(G)-\{r\}} C(v)\mathsf{w}_i(v) \leq \sum_{i=1}^k \sum_{v \in V(G)-\{r\}} \mathsf{w}_i(v),$$

$$\sum_{v \in V(G)-\{r\}} C(v) \sum_{i=1}^k \mathsf{w}_i(v) \leq \sum_{v \in V(G)-\{r\}} \sum_{i=1}^k \mathsf{w}_i(v), \text{ and}$$

$$c|C| \leq c(n-1),$$

for all r-unsolvable C, from which follows $|C| \leq n - 1$. Hence if $|C| \geq n$ then C is r-solvable. □

Exercises.

(1) The path P_n on n vertices has rooted pebbling number $\pi(P_n, r) = 2^{n-1}$ when r is one of its leaves.

(2) Let P be the Petersen graph and r be any of its vertices. Prove (without using Lemma 1.8 or its consequences) that $\pi(P, r) = 10$.

(3) Every graph G on n vertices has rooted pebbling number $\pi(G,r) \leq (n-1)(2^{\mathsf{ecc}_G(r)}-1)+1$ for every root vertex r.
(4) Use induction to prove Theorem 1.1.
(5) [19] Let G be a graph in which each of its blocks is a clique, and suppose that T is a breadth-first search spanning tree of G rooted at r. Then $\pi(G,r)=\pi(T,r)$.
(6) Prove Lemma 1.2 by considering minimal solutions.
(7) Find the largest c for which there is a configuration on the cycle C_n of size at least $c\pi(C_n,r)$ that is r-solvable but not r-semi-greedy.
(8) Prove Lemma 1.3.
(9) Use Lemma 1.3 to prove Theorem 1.4.
(10) Prove Theorem 1.7.
(11) Use induction to prove Lemma 1.8.
(12) Prove that every r-strategy is a conic combination of basic r-strategies. That is, for every r-strategy (T,w) of a rooted graph (G,r), there are basic r-strategies $(T_1,\mathsf{w}_1),\ldots,(T_h,\mathsf{w}_h)$ of (G,r) and nonnegative coefficients $\alpha_1,\ldots,\alpha_h$ so that, for all $v\in v(G)$, we have $\mathsf{w}(v)=\sum_{i=1}^{h}\mathsf{w}_i(v)$.
(13) Use Proposition 1.9 to prove Theorem 1.1. (Hint: Relate Figures 2 and 4 to each other.)
(14) Let P be the Petersen graph and r be any of its vertices. Use Lemma 1.10 to prove that $\pi(P,r)=10$.
(15) Let L be the Lemke graph and r be any of its vertices. Use Lemma 1.8 to prove that $\pi(L,r)=8$.
(16) Use Lemma 1.8 to prove Theorem 1.4.
(17) (Open) Is there a characterization for r-solvable configurations on trees rooted at r?
(18) (Open) Is $\pi_{\mathbf{T}}(G,r)\leq 2\pi(G,r)$ for every rooted graph (G,r)?
(19) (Open) Find larger classes of strategies than those arising from trees.

2. Pebbling Numbers

We turn our attention now to configurations that are r-solvable for every possible root r.

We say that a configuration C on G is *(k-fold) solvable* if it is (k-fold) r-solvable for every vertex r. The *k-fold pebbling number* $\pi_k(G)$ is defined to be the minimum number t so that every configuration C on G of size t is k-fold solvable. In other words, $\pi_k(G)=\max_r \pi_k(G,r)$. (We write $pi(G)=\pi_1(G)$.) From the prior section, then, we have $\pi(K_n)=n$, $\pi(P_n)=2^{n-1}$, $\pi(C_{2k})=2^k$ for all $k\geq 2$, $\pi(C_{2k+1})=\lceil(2^{k+2}-1)/3\rceil$ for all $k\geq 1$, and $\pi(P)=10$.

In this realm the version of the Distance Bound becomes $\pi(G)\geq 2^{\mathsf{diam}(G)}$. Interestingly, both the Vertex and Distance Bounds give $\pi(Q^d)\geq 2^d$. In the very first graph pebbling paper, Chung proved the important result (below) that this bound is tight. The result is important because of its number theoretic application.

2.1. Graph Products.

For two graphs G_1 and G_2, define the *cartesian product* $G_1\square G_2$ to be the graph with vertex set $V(G_1\square G_2)=\{(v_1,v_2)\mid v_1\in V(G_1),v_2\in V(G_2)\}$ and edge set $E(G_1\square G_2)=\{\{(v_1,v_2),(w_1,w_2)\}\mid(v_1=w_1 \text{ and } (v_2,w_2)\in E(G_2)) \text{ or } (v_2=w_2 \text{ and } (v_1,w_1)\in E(G_1))\}$. We write $\Pi_{i=1}^k G_i$ to mean $G_1\square\ldots\square G_k$ and set $G^k=\Pi_{i=1}^k G$. Thus $Q^d=P_2^d$.

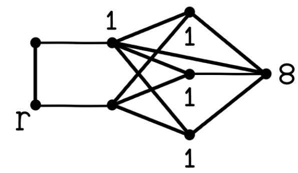

FIGURE 7. The Lemke graph with a 2-fold r-unsolvable configuration.

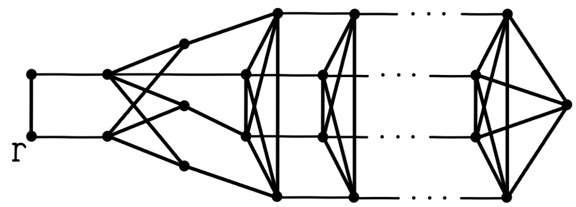

FIGURE 8. An infinite family of rooted Lemke graphs.

Theorem 2.1. [15] *The d-dimensional cube Q^d has $\pi(Q^d) = 2^d$. More generally, let $G = \Pi_{i=1}^k P_{l_i+1}$ be the cartesian product of k paths of lengths $l_i = \mathsf{diam}(P_{l_i+1})$, with $l = \mathsf{diam}(G) = \sum_{i=1}^k l_i$. Then $\pi(G) = 2^l$.*

The proof of this result depends on an interesting property involving 2-fold solvability. In general, for configurations C with $C(r) = 0$, solving r is equivalent to 2-fold solving some neighbor of r. Certainly, $\pi_2(G) \leq 2\pi(G)$: pretend that half of the $2\pi(G)$ pebbles are red and half are blue; then each color provides a solution. But fewer pebbles might suffice: after the red solution there may be residual red pebbles that could be used for the blue solution. Thus, instead of measuring a configuration by size only, we include a bit of its structure.

The *support* $\mathsf{supp}(C)$ of a configuration C on G is the set of vertices that have a pebble of C; i.e. $\mathsf{supp}(C) = \{v \in V(G) \mid C(v) > 0\}$. The size of the support is denoted $s(C) = |\mathsf{supp}(C)|$. A graph G has the *2-pebbling property* (2PP) if every configuration C of size at least $2\pi(G) - s(C) + 1$ is 2-fold solvable. For example, the complete graph K_n has the 2-pebbling property because the maximum number of pebbles that can be placed on s vertices without having either, in the case of some vertex being empty, two vertices with at least two pebbles or one vertex with at least four pebbles is $s + 2$ or, in the case of no vertex being empty, one vertex with at least two pebbles is s, which in both cases is stricly less than $2n - s + 1$.

Lemke discovered the graph L in Figure 7. It is the smallest graph that does not have 2PP: we have $\pi(L) = 8$ and $|C| = 12 = 2(8) - 5 + 1$, but C cannot place two pebbles on r. Any graph not satisfying 2PP is called *a Lemke* graph; L is *the Lemke* graph. Wang [75] was the first to find infinitely many graphs (see Figure 8) without 2PP. His sequence was based on an earlier sequence conjectured by Foster and Snevily [30].

The *odd support* $\mathsf{supp}^o(C)$ of a configuration C is the set of vertices that have an odd number of pebbles of C. The size of the odd support is denoted $s^o(C) = |\mathsf{supp}^o(C)|$. A graph G has the *odd 2-pebbling property* (O2PP) if every configuration C of size at least $2\pi(G) - s^o(C) + 1$ is 2-fold solvable.

The key to Chung's proof that

$$\pi(Q^d) = 2^d \tag{2.1}$$

is to prove along with it that

$$Q^d \text{ has } \mathsf{O2PP}. \tag{2.2}$$

This is done simultaneously by induction, as each is trivially true when $d = 0$. We split Q^d naturally into two copies, Q_1 and Q_2, each isomorphic to Q^{d-1}, with $r \in Q_1$ and its neighbors $r' \in Q_2$. The restriction C_i of C on Q_i has size $|C_i| = t_i$, and $\sigma^o(C_i) = \sigma_i$.

First we prove (2.1). Of course, if $t_1 \geq 2^{d-1}$ the C_1 is r-solvable by induction, so we assume otherwise. If $\sigma_2 > t_1$ then $t_2 = 2^d - t_1 > 2\pi(Q_2) - \sigma_2$ and so, by induction on (2.2), C_2 is 2-fold r'-solvable, and hence r-solvable. Thus we may assume that $\sigma_2 \leq t_1$. In this case we can move $(t_2 - \sigma_2)/2$ pebbles from Q_2 to Q_1, resulting in $t_1 + (t_2 - \sigma_2)/2 \geq t_1 + (t_2 - t_1)/2 = (t1 + t_2)/2 = 2^{d-1} = \pi(Q_1)$ pebbles in Q_1, which means one pebble can be moved to r from these.

Next we prove (2.2); now we have $t_1 + t_2 = 2^{d+1} - \sigma_1 - \sigma_2 + 1$. Of course, if $t_1 \geq 2^d - \sigma_1 + 1$ then C_1 is 2-fold r-solvable. If $2^{d-1} \leq t_1 \leq 2^d - \sigma_1$, then C_1 is r-solvable and C_2 is 2-fold r'-solvable (see exercises). If $t_1 < 2^{d-1}$ then the trick is to move the right number of pebbles from Q_2 to Q_1 so that the resulting number of pebbles in Q_1 is r-solvable while the remaining number of pebbles in Q_2 is 2-fold r'-solvable (see exercises).

Chung's result gave rise to the following conjecture, which many feel is the Holy Grail of the subject.

Conjecture 2.2. [Graham's Conjecture] *Every pair of graphs G_1 and G_2 satisfy $\pi(G_1 \square G_2) \leq \pi(G_1)\pi(G_2)$.*

Many instances of this conjecture have been verified, including when G_1 and G_2 are both cycles ([30, 40]) or both trees ([15, 30]), and when G_1 is a tree, cycle, complete graph, or complete bipartite graph and G_2 has 2PP ([15, 64, 41]). The latter instance has provided motivation for studying 2PP, giving rise to results like the following.

Proposition 2.3. [67] *If $\mathsf{diam}(G) = 2$ then G has* 2PP.

Proposition 2.4. [34] *If G is a bipartite graph with largest part size $s \geq 15$ and minimum degree at least $\lceil \frac{s+1}{2} \rceil$ then $\pi(G) = n$ and G has* 2PP.

The techniques of the next subsection can be used to prove Theorem 2.5.

Theorem 2.5. [26] *Graham's Conjecture is satisfied when $\delta(G_i) \geq k \geq 2^{12n/k+15}$.*

But hold your horses; Hurlbert [52] suggests that L^2 might be a counterexample to Graham's Conjecture.

2.2. **Diameter, Connectivity, and Class 0.** In this subsection we study graphs having smallest possible pebbling number. We say that a graph G is of *Class 0* if $\pi(G) = n$. Examples of Class 0 graphs we have seen so far include K_n, P, L, Q^d, C_5^d, and bipartite graphs from Proposition 2.4. One way of constructing new Class 0 graphs from old ones, if Graham's Conjecture were true, would be via cartesian product. Another way comes from the following type of graph sum. Let $G_1 + G_2$ denote the disjoint union of G_1 and G_2.

Theorem 2.6. [49] *Given Class 0 graphs G_1 and G_2 let F be any bipartite subgraph of $V(G_1) \times V(G_2)$ with no isolated vertices. Then $(G_1 + G_2) \cup F$ is Class 0.*

Note that this generalizes Theorem 2.1 because $Q^{d+1} = (Q^d + Q^d) \cup F$, where F is a perfect matching.

The first considerations of fixed diameter graphs came from a result of Pachter, Snevily and Voxman.

Theorem 2.7. [67] *If* $\mathsf{diam}(G) = 2$ *then* $\pi(G) \le n + 1$.

Proof. Given any configuration C on G, define V_i to be the set of vertices having i pebbles, with $n_i = |V_i|$. Then $n = \sum_{i \ge 0} n_i$ and $|C| = \sum_{i \ge 0} i n_i$. Of course, since $\mathsf{diam}(G) = 2$ we have $n_i = 0$ for $i \ge 4$. Assume that C is r-unsolvable of size $n + 1$.

G being diameter two means that nonadjacent vertices have a common neighbor, and C being r-unsolvable means that r has no big neighbors and that no two big vertices share a common neighbor with r. Hence the number of empty neighbors of r is at least $n_2 + n_3$; i.e. $n_0 \ge 1 + n_2 + n_3$. This implies that $n \ge n_1 + 2n_2 + 2n_3 = |C| - n_3$, so that $n_3 \ge |C| - n = 1$. Let u be a vertex with 3 pebbles.

Now there are at least another $n_2 + n_3 - 1$ distinct empty vertices that are common neighbors of u and other big vertices — otherwise we could move a fourth pebble to u from another big vertex. Thus $n_0 \ge 2n_2 + 2n_3$, and so $n \ge n_1 + 3n_2 + 3n_3 = |C| + n_2 > n$, a contradiction. Hence C is r-solvable. □

More generally, Herscovici, et. al, prove the following.

Theorem 2.8. [45] *If* $\mathsf{diam}(G) = 2$ *then* $\pi_k(G) \le n + 4k - 3$.

The history of the term Class 0 comes from Theorem 2.7 — diameter two graphs come in two classes: $\pi = n$ or $n + 1$. In fact, it is possible to characterize which is which. The *pyramid* is any graph on 6 vertices isomorphic to the union of the 6-cycle (r, a, p, c, q, b) and the (inner) triangle (a, b, c). A *near-pyramid* is a pyramid minus one of the edges of its inner triangle (a, b, c). A graph G is *pyramidal* if it contains an induced (near-) pyramid, having 6-cycle C and inner (near-) triangle K, and can be drawn in the plane so that

(1) the edges of K are drawn in the interior of the region bounded by C and
(2) every other edge of G can be drawn inside the convex hull of exactly one of the sets $\{r, a, b\}$, $\{p, a, c\}$, $\{q, b, c\}$, or $\{a, b, c\}$.

Note that a pyramidal graph is not Class 0 (see exercises).

Theorem 2.9. [16] *If* $\mathsf{diam}(G) = 2$ *and* $\kappa(G) \ge 2$ *then G is Class 1 if and only if G is pyramidal.*

The role of connectivity here comes from the fact that no graph with a cut vertex is Class 0 (see exercises). One can use the same arguments from the proof of Theorem 2.7, but with an r-unsolvable size n configuration, to show that it must have $t_3 = 2$, which gives rise to the 6-cycle (we also learn that $t_2 = 0$); the inner triangle is forced by diameter and other V_1 vertices can be added carefully as described. Because pyramidal graphs are not 3-connected, we find the following corollary.

Corollary 2.10. [16] *If* $\mathsf{diam}(G) = 2$ *and* $\kappa(G) \ge 3$ *then G is Class 0.*

This corollary immediately suggests that there may be a strong relationship between diameter, connectivity, and Class 0. Indeed, Czygrinow, et al., prove this so.

Theorem 2.11. [26] *There is a function* $k(d) \le 2^{2d+3}$ *such that if G is a graph with* $\mathsf{diam}(G) = d$ *and* $\kappa(G) \ge k(d)$ *then G is of Class 0. Moreover,* $k(d) \ge 2^d/d$.

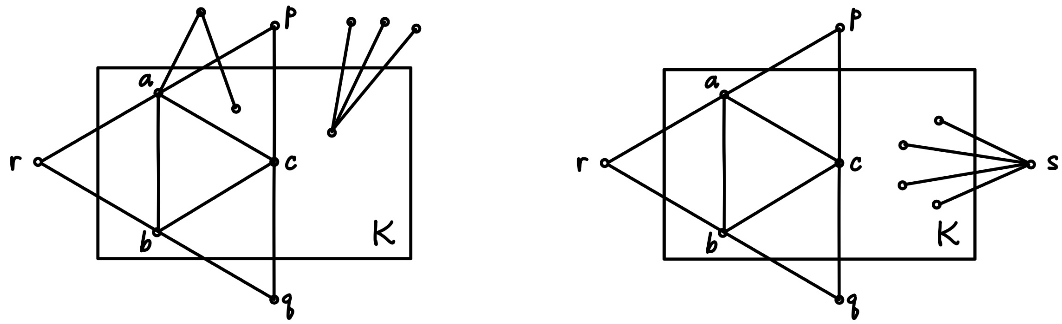

FIGURE 9. Examples of Pereyra (left) and Phoenix (right) graphs

Theorem 2.11 has some powerful consequences. First, it verifies Graham's Conjecture for dense graphs.

Theorem 2.12. [23] *There is a constant c so that if $\delta(G_i) > cn/\lg n$ for $i \in \{1,2\}$ then $G_1 \square G_2$ is Class 0.*

Second, it shows that most Kneser graphs are Class 0. For $m \geq 2t+1$ the *Kneser* graph $K(m,t)$ has as vertices all t-subsets of $\{1,2,\ldots,m\}$ and edges between every pair of disjoint sets. For example $K_n = K(n,1)$ and $P = K(5,2)$.

Theorem 2.13. [26] *For any constant $c > 0$ there is an integer t_0 such that, for $t > t_0$, $s \geq c(t/\lg_2 t)^{1/2}$ and $m = 2t+s$, we have $\kappa(K(m,t)) \geq 2^{2d+3}$, where $d = \mathsf{diam}(K(m,t))$; hence $K(m,t)$ is Class 0.*

Third, note that almost all graphs are Class 0. Indeed, choosing from graphs with edge probability 1/2, the probability that some two vertices have no common neighbor is at most $n^2(3/4)^n \to 0$, and so almost all graphs have diameter two. Similarly, they are almost surely 2-connected and have no prescribed structure like being pyramidal (see exercises). Thus Theorem 2.9 applies. But Theorem 2.11 implies that even very sparse graphs are almost all Class 0.

Theorem 2.14. [26] *Let $G \in \mathcal{G}(n,p)$ be a random graph on n vertices with edge probability p and let $d = \mathsf{diam}(G)$. If $p \gg (n \lg_2 n)^{1/d}/n$ then $\mathbf{Pr}[\kappa(G) \geq 2^{2d+3}] \to 1$ as $n \to \infty$; hence $\mathbf{Pr}[G$ is Class 0$] \to 1$ as $n \to \infty$.*

Fourth and finally, it implies the lower bound in the following theorem of Czygrinow and Hurlbert. Let $g_0(n)$ denote the maximum number g such that there exists a Class 0 graph G on at most n vertices with finite $\mathsf{gir}(G) \geq g$.

Theorem 2.15. [24] *For all $n \geq 3$ we have*

$$\lfloor \sqrt{(\lg_2 n)/2 + 1/4} - 1/2 \rfloor \leq g_0(n) \leq 1 + 2\lg_2 n.$$

Cycles deliver the upper bound, but the lower bound depends on a result of Bollobas that guarantees the existence of a graph G of large girth g and large minimum degree δ on at most $(2\delta)^g$ vertices, a result of Mader that shows that G has a subgraph of connectivity at least $\lfloor \delta/4 \rfloor$, and then an application of Theorem 2.11.

We turn to a few results involving graphs of diameter 3 or 4. A graph G is a *split* graph if its vertices can be partitioned into a clique K and an independent set I of *cone* vertices. For $n = 2k(+1)$, the *sun* S_n is the split graph with perfect matching joining $I = I_k$ to $K = K_k$ (and one extra leaf when n is odd). A split graph is *Pereyra* if it has a pyramid, none of whose vertices is a cut vertex, and a Pereyra graph is *Phoenix* if some corner of its pyramid has a vertex of degree at least 4 at distance 3 from it (see Figure 9).

The following result of Postle, et al., is the first in pebbling to use discharging-type techniques. Given a graph G, an r-unsolvable configuration C of size matching the upper bound, and a breadth-first search spanning tree T, the partition T uniquely into what they call irreducible branches (subtrees) with pebbling capacity 0, which roughly means that the parts are the smallest substructures that emit no pebbles towards r. The *charge* of a set W of vertices equals $\sum_{v\in W}(C(v) - 3/2)$, and W is called *superoptimal* if it has positive charge and *suboptimal* otherwise. Note that $V(G)$ has positive charge so, by averaging, at least one of the branches in the partition must be superoptimal. They then produce a list of all 12 irreducible superoptimal branches of capacity 0 — this is the unavoidable set. Discharging arguments traditionally proceed by showing that in a minimal counterexample the unavoidable set consists of reducible configurations, which then contradicts minimality. Here, though, without minimality in use, Postle, et al., find a contradiction by very carefully and technically matching irreducible superoptimal capacity 0 branches with suboptimal branches that cancel their charge, giving total nonpositive charge for $V(G)$.

Theorem 2.16. [71] *If* $\mathsf{diam}(G) = 3$ *then* $\pi(G) \le \lfloor 3n/2 \rfloor + 2$, *which is best possible, as shown by the sun* S_n. *If* $\mathsf{diam}(G) = 4$ *then* $\pi(G) \le 3n/2 + c$, *for some constant* c.

They use discharging more traditionally, complete with discharging rules, to prove that if $\mathsf{diam}(G) = d$ then $\pi(G) \le (2^{\lfloor d/2 \rfloor} - 1)n + 2^{4d} + 1$.

Alcón, et al., gave the first account of exact pebbling numbers for an infinite family of diameter 3 graphs.

Theorem 2.17. [1] *If* G *is a diameter 3 split graph then* $\pi(G)$ *is given as follows. Let* x *be the number of cut vertices of* G *and, for a vertex* r, *define* $\delta^* = \delta^*(G, r)$ *to be the minimum degree of a vertex at maximum distance from* r.

(1) *If* $\mathrm{x} \ge 2$ *then*
$$\pi(G) = n + \mathrm{x} + 2.$$
(2) *If* $\mathrm{x} = 1$ *then*
$$\pi(G) = \begin{cases} n + 5 - \delta^* & \text{if } r \text{ is a leaf with } \mathsf{ecc}(r) = 3 \text{ and } \delta^* \le 4; \\ n + 1 & \text{otherwise.} \end{cases}$$
(3) *If* $\mathrm{x} = 0$ *then*
$$\pi(G) = \begin{cases} n + 4 - \delta^* & \text{if there is a cone vertex } r \text{ with } \deg(r) = 2, \\ & \quad \mathsf{ecc}(r) = 3, \text{ and } \delta^* \le 3; \\ n + 1 & \text{if no such } r \text{ exists and } G \text{ is Pereyra;} \\ n & \text{otherwise.} \end{cases}$$

Corollary 2.18. [1] *If* G *is a split graph with* $\delta(G) \ge 3$ *then* G *is Class 0.*

Of course this implies that 3-connected split graphs are Class 0. We finish this section with an interesting variation on the Class 0 theme. The k^{th} *graph power* $G^{(k)}$ of a graph G is formed from G by adding edges between every pair of vertices of distance at most k in G. Pachter, et al. [67], define the *pebbling exponent* $e_\pi(G)$ of a graph G to be the minimum k such that $G^{(k)}$ is Class 0. The Distance Bound implies that $e_\pi(G) \ge \frac{n/2}{\lg n}$. Hurlbert uses the weight function method to prove the following near tight upper bound for cycles.

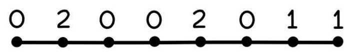

FIGURE 10. A minimum solvable configuration on the path P_8.

Proposition 2.19. [52] *For every $n \geq 3$ we have*

$$e_\pi(C_n) \leq \frac{n/2}{\lg n - \lg\lg n}.$$

Exercises.

(1) [15] Let r^* be a leaf of a longest path in a tree T. Prove that $\pi(T) = \pi(T, r^*)$.
(2) Show that P_n satisfies 2PP.
(3) Show that Q^d satisfies 2PP.
(4) Complete the proof of (2.2) in the case that $2^{d-1} \leq t_1 \leq 2^d - \sigma_1$.
(5) Complete the proof of (2.2) in the case that $t_1 < 2^{d-1}$.
(6) Use Theorem 2.6 to prove that P is Class 0.
(7) Prove that $\kappa(G) = 1$ implies $\pi(G) > n$.
(8) Prove that if G is pyramidal then $\pi(G) > n$.
(9) Prove that the function k from Theorem 2.11 satisfies $k(d) \geq 2^d/d$. (HINT: Consider a kind of "fattened" path.)
(10) Use Theorem 2.11 to prove Theorem 2.13.
(11) Prove that the probability that a random graph in $\mathcal{G}(n, 1/2)$ is pyramidal is less than $n^6/2^{3n}$.
(12) Use Theorem 2.11 to prove Theorem 2.14.
(13) If G is a graph on n vertices with $\mathsf{diam}(G) = d$ then $e_\pi(G) \geq d/\lg n$.
(14) Prove that every graph with at least $\binom{n-1}{2} + 2$ edges is Class 0. Show that this bound is best possible.
(15) [8] Prove that every Class 0 graph has at least $\lfloor 3n/2 \rfloor$ edges.
(16) (Open) Does every bipartite graph have the 2-pebbling property?
(17) (Open) Is $\pi(L^2) = 64$?
(18) (Open) Find infinitely many Class 0 graphs with n vertices and at most $3n/2 + o(n)$ edges.
(19) (Open) Decide if $K(m, t)$ is Class 0 for all $m = 2t + s$ with $s \in O((t/\lg_2 t)^{1/2})$.
(20) (Open) Find the smallest $k(d)$ such that G is Class 0 for every diameter d graph G with $\kappa(G) \geq k(d)$.

3. OPTIMAL PEBBLING

While pebbling can be thought of as a worst-case scenario — we give an adversary enough pebbles so that we can solve the graph no matter how she arranges them — optimal pebbling can be considered a best-case scenario — we place few pebbles carefully so as to solve the graph.

3.1. **Basics and Smoothing.** The *optimal pebbling number* $\pi^*(G)$ is the minimum number t for which there exists a solvable configuration of size t. For example, the configuration with 2 pebbles on a single vertex can reach any other vertex of the complete graph, and so $\pi^*(K_n) = 2$ for all n. Figure 10 displays the upper bound of $\pi^*(P_8) \leq \lceil 2(8)/3 \rceil$; in general, it is easy to see that every graph G satisfies $\pi^*(G) \leq 2\mathsf{dom}(G)$, since placing 2 pebbles on each vertex of a minimum dominating set yields a solvable configuration. Moreover, call a set of vertices W a *distance k dominating set* of G if every vertex v is at distance at most k from some vertex of W, and define $\mathsf{dom}_k(G)$ to be the minimum size of such a set. Then the following is immediate.

Proposition 3.1. *Every graph G satisfies $\pi^*(G) \le 2^k \mathsf{dom}_k(G)$ for every $k \ge 1$.*

This implies that $\pi^*(G) \le 2^{\mathsf{rad}(G)}$ since any center vertex is a distance $\mathsf{rad}(G)$ dominating set. Moews [65] uses this Proposition to show that the d-cube has $\pi^*(Q^d) \le (4/3)^d d^2$. Choosing $k = \lceil d/3 \rceil$, he uses a result of Cohen that says that $\mathsf{dom}_k(Q^d) \le 2^{sd} d^2$, where $s = (5/3 - \lg 3)$. Moews also shows that the bound is nearly tight: $\pi^*(Q^d) \ge (4/3)^d$. The following argument for this is due to Bunde, et al. [13]. Suppose that C is solvable. Then Corollary 1.6 holds for every r; that is, $\sum_{k=0}^{d} C_k(r)/2^k \ge 1$, where $C_k(r)$ denotes the number of pebbles at distance k from r. Now choose r uniformly at random. Then $\mathbf{E}[C_k(r)] = |C|\binom{d}{k}/2^d$, so that applying linearity of expectation to Corollary 1.6 yields $|C|(3/4)^d = |C|(1+1/2)^d/2^d = |D| \sum_k \binom{d}{k} 2^{-k}/2^d \ge 1$.

Because the upper bound is generated nonconstructively, there have been attempts at producing constructive bounds. Pachter, et al. [67] give an $O(\sqrt{2}^d)$ construction by placing pebbles at a pair of antipodal vertices. Herscovici, et al. [43] give an iterative contruction that yields an $O(c^d)$ bound, where $c \approx 1.3763$.

The Subgraph Bound applies to optimal pebbling as well — $\pi^*(G) \le \pi^*(H)$ when H is a spanning subgraph of G — and for the same reason: adding edges can't destroy the solvability of a configuration. The version of the Distance Bound for optimal pebbling is $\pi^*(G) \ge \lceil 2(d+1)/3 \rceil$, where $d = \mathsf{diam}(G)$ (see exercises), but the Vertex Bound in this case is an upper bound. The configuration $\mathbf{J}$ (one pebble on each vertex) shows that $\pi^*(G) \le n$, but a stronger bound is known.

Theorem 3.2. [13] *For all G we have $\pi^*(G) \le \lceil 2n/3 \rceil$.*

Before we can prove this we need to introduce a key technique called smoothing, sort of the opposite of squishing. Let C be a configuration on G and suppose that $\deg(v) = 2$ and $C(v) \ge 3$. A *smoothing move* at v removes two pebbles from v and adds one pebble to each of its neighbors. One can prove (see exercises) that, if C' is the configuration obtained by making a smoothing move from an r-solvable configuration C, then C' is also r-solvable. A *smooth* configuration has no smoothing move available; that is, C is smooth if $C(v) \le 2$ whenever v has degree 2.

Lemma 3.3. [Smoothing Lemma] [13] *If G has at least 3 vertices then G has a smooth minimum sized solvable configuration with empty leaves.*

Proof. The proof begins by noting that an arbitrary solvable configuration C can be assumed to have minimum size, at most $\lceil 2n/3 \rceil$ (and so has an empty vertex), then showing that only finitely many smoothing moves can be made from C. Indeed, if G is not a cycle, consider a thread T contained in the unique smallest path P_k having endpoints of degree greater than 2. For a pebble on T give it the weight $d(k-d)$ when it has distance d from an endpoint of P_k. Then a smoothing move reduces the sum of the weights of the two pebbles involved by 2. Since the sum of the weights of the pebbles on T must remain nonnegative, the thread will be smooth in a finite number of moves. If G is a cycle then let v be an empty vertex of C and use the same weight argument as above by considering v as both endpoints of a path. Repeated smoothing moves will either create a smooth configuration or move pebbles onto v, which decreases the number of empty vertices. Thus only finitely many smoothing moves are possible.

Now that C is smooth, we must create empty leaves. When a leaf v has neighbor u with $C(v)+C(u) \geq 3$ pebbles we can move $C(v)-1$ of them to u and remove the other to create the contradiction of a smaller solvable configuration. Otherwise we move all of the pebbles from v to u to achieve the desired result (see exercises). □

The bound in Theorem 3.2 is best possible, as shown by paths and cycles.

Theorem 3.4. [13, 31, 67] *For all $n \geq 3$ we have $\pi^*(P_n) = \pi^*(C_n) = \lceil 2n/3 \rceil$.*

Proof. We prove this for P_n only and leave C_n for the exercises. Because of Proposition 3.1 we only need to prove the lower bound, which we do by induction, assuming that $n \geq 7$ because smaller cases can be shown by hand (see exercises).

Let C be a minimum sized solvable configuration. Since $|C| \leq \lceil 2n/3 \rceil$ there must be at least three empty vertices; let v be one that is not an endpoint of P_n and denote by P_n^L and P_n^R the two paths whose union is P_n and intersection is $\{v\}$.

The generalization of Corollary 1.6 for 2-fold v-solvability on each of P_n^L and P_n^R requires weight at least two, which is impossible since C is smooth. Thus the No-Cycle Lemma implies that any r-solution from C happens locally on either P_n^L or P_n^R. By induction, with $P_n^L \cong P_a$ and $P_n^R \cong P_b$, we have $|C| \geq \lceil 2a/3 \rceil + \lceil 2b/3 \rceil \geq \lceil 2n/3 \rceil$. □

As a step toward proving Theorem 3.2 we show that the bound holds for trees.

Theorem 3.5. [13] *If T is a tree then $\pi^*(T) \leq \lceil 2n/3 \rceil$.*

Proof. The inductive proof when $n > 3$ finds a set W of at least three vertices to remove so that two well placed pebbles on some vertex of W can reach every other vertex of W either alone or with the help of the appropriate solution on $T' = T - W$. Let x be a leaf of a longest path P in T and y be its neighbor. The first three cases to consider are when $\deg(y) > 2$, when $\deg(y) = 2$ and its other neighbor z has $\deg(z) = 2$, and when $\deg(y) = 2$, $\deg(z) > 2$, and z has a leaf neighbor w; we leave these cases to the exercises.

Now let v be a neighbor of z not on P, let u be a leaf neighbor of v, and let $W = \{x, y, u\}$. Write C' for a minimum sized solvable configuration on T'. If C' is 2-fold z-solvable then form C from C' by adding two pebbles to z. Then C is solvable because 4 pebbles can reach vertices within distance two. Otherwise we form C from C' by adding two pebbles to y. If C' is 2-fold w-solvable then those two pebbles can be used to solve v. Otherwise the edge $\{z, w\}$ is not used in any solution and so C' is simultaneously z-solvable and w-solvable, so the two pebbles from y can solve v. □

3.2. **Collapsing and Minimum Degree.** One might improve on the 2/3 coefficient in Theorem 3.2 by considering minimum degree $\delta = \delta(G)$. Let $N_k[v]$ denote the closed k-neighborhood of a vertex v; i.e. the set of all vertices in G at distance at most k from v. Denote by $n_k(G)$ be the minimum size of $N_k[v]$ over all v. Czygrinow gave an argument in [13] that $\mathsf{dom}_{2k}(G) \leq n/n_k(G)$. In the case $k = 1$ we have $n_1(G) = \delta + 1$, which, combined with Proposition 3.1 yields the following improvement on Theorem 3.2 when $\delta \geq 6$.

Theorem 3.6. [13] *For all G we have $\pi^*(G) \leq \frac{4n}{\delta+1}$.*

The complete graph K_n shows that there exist graphs with $\pi^*(G) \geq 2n/(\delta+1)$. Here $\delta = n + 1$ but, in fact, more examples abound, even with $\delta \ll n$. For a vertex v in a graph G we define the *t-blow up* of G at v to have vertices $(V(G) - \{v\}) \cup \{v_1', \ldots, v_t'\}$ with edges $(E(G) - \{uv \mid u \in V(G)\}) \cup \{uv_i' \mid uv \in E(G), 1 \leq i \leq t\} \cup \{v_i'v_j' \mid 1 \leq$

$i < j \le t\}$. That is, v is replaced with t clones of it that form a t-clique. Now let the vertices of C_{3s} be labeled $v_1, \ldots, v_{3s}$ around the cycle, and define the graph $G_{s,t}$ to be formed from C_{3s} by t-blowing up of each of the vertices v_{3i}. It turns out (see exercises) that $\pi^*(G_{s,t}) = \pi^*(C_{3s}) = 2s = 2n/(\delta+1)$.

The following stronger result is proven in [13].

Theorem 3.7. [13] *For all $t \ge 1$ there is a graph G with $\delta = 3t \le n-3$ and $\pi^*(G) \ge (2.4 - \frac{24}{5\delta+15} - o(1))\frac{n}{\delta+1}$.*

The proofs of these results depend on the technique of collapsing, which is a little bit like the reverse of blowing up but without requiring cliques. In fact, it's more like contracting but without requiring the collapsing set to be connected. For $W \subseteq V(G)$, the operation of *collapsing* W forms a new graph H in which W is replaced by a single vertex w that is adjacent to all the neighbors of vertices of W that are in $V - W$.

Lemma 3.8. [Collapsing Lemma] [13] *If H is obtained from G by collapsing sets of vertices then $\pi^*(G) \ge \pi^*(H)$.*

Proof. Let C be a solvable configuration on G and let H be formed from G by collapsing W to the new vertex w. Define C' on H by $C'(w) = C(W)$ and $C'(v) = C(v)$ for all $v \ne w$. Now any pebbling step on the edge e in G naturally collapses to a pebbling step in H by being the same step if $e \in E(H)$ and by being a non-step if $e \notin E(H)$. Thus any sequence of pebbling steps that solves some root $r \in V(G) - W$ still solves $r \in V(H)$, and that solves some root $r \in W$ now solves w. Thus C' is solvable on H. □

Exercises.

(1) Prove that if $d = \mathsf{diam}(G)$ then $\pi^*(G) \ge \lceil 2(d+1)/3 \rceil$.
(2) [66] Exercise 1 and Proposition 3.1 imply that $2 \le \pi^*(G) \le 4$ if $\mathsf{diam}(G) = 2$. Characterize when $\pi^*(G) = 2$ and when $\pi^*(G) = 3$.
(3) Let C' be the configuration obtained by making a smoothing move from an r-solvable configuration C. Prove that C' is also r-solvable.
(4) Complete the proof of Lemma 3.3 by emptying the leaves.
(5) Prove that $\pi^*(P_n) = \lceil 2n/3 \rceil$ for $n \le 6$.
(6) Prove that $\pi^*(C_n) = \lceil 2n/3 \rceil$ for $n \ge 3$.
(7) Prove Theorem 3.5.
(8) Use Theorem 3.5 to prove Theorem 3.2.
(9) Prove that if G has girth $2t+1$ then $\pi^*(G) \le 2^{2t}/(1 + \delta \sum_{i=1}^{t} (\delta-1)^i)$.
(10) Use Lemma 3.8 to prove that the graph $G_{s,t}$, defined after Theorem 3.6, satisfies $\pi^*(G_{s,t}) = 2n/(\delta+1)$.
(11) Define the *Sierpinski graphs* S_m as follows (see Figure 11): $S_1 = K_3$ with corner vertices $\{x, y, z\}$ and, for $m > 1$, S_m is built from three copies of S_{m-1} with corner vertices $\{x_i, y_i, z_i\}$ by collapsing the pairs of vertices $\{y_1, x_2\}$, $\{z_2, y_3\}$, and $\{x_3, z_1\}$, creating the new corner vertices $\{x_1, y_2, z_3\}$. (For example, S_2 is the pyramid.) Now define the graph R_m by adding three edges between the corner vertices. Prove that $\pi^*(R_m) \le 2(3^{m-3})$.
(12) For all $2 \le \delta < n$ find graphs G having $\pi^*(G) > 2n/(\delta+1) - 2$.
(13) [32, 43] For all graphs G and H we have $\pi^*(G \square H) \le \pi^*(G)\pi^*(H)$.
(14) Find graphs G with maximum degree less than $\lg n$ and $\pi^*(G) \ll n$.
(15) (Open) Characterize those graphs G for which $\pi^*(G) = 2\mathsf{dom}(G)$.
(16) (Open) Is there a graph G with $\pi^*(G) \ge 3n(G)/(\delta(G)+1)$?

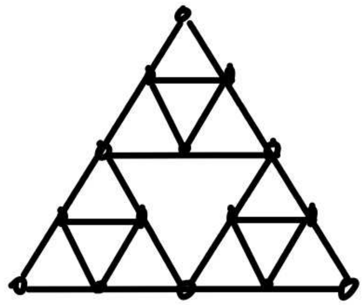

FIGURE 11. The Sierpinski graph S_3.

(17) (Open) Does $\delta(G) \geq 3$ imply that $\pi^*(G) \leq \lceil n(G)/2 \rceil$?

4. Complexity

Here we discuss questions such as how long it takes to decide if a particular configuration C on a graph G is D-solvable, to give an upper bound on $\pi(G, r)$ for a rooted graph (G, r), or to calculate $\pi^*(G)$, for example. Unless otherwise stated, running times of algorithms will be in terms of n.

4.1. **Solvability.** We begin by defining D-**SOLVABLE** (resp., **SOLVABLE**) to be the problem of deciding, for a configuration C on a graph G, if C is D-solvable (resp., solvable). For example, if $G = P_n$ and r is a leaf, we know that C is r-solvable if and only if $wt(C) \geq 1$, which can be checked in linear time. However, the problem is more difficult in general, as discovered by Hurlbert and Kierstead [56] (see also [59]). First we note that r-SOLVABLE $\in$ NP.

Theorem 4.1. [59] *The configuration C is D-*solvable *on G if and only if there is a nonnegative integral solution to the system*

$$\{C(u) + \sum_{v \in V(G)-\{u\}} (x_{v,u} - 2x_{u,v}) \geq D(u)\}_{u \in V}. \tag{4.1}$$

Hence D-SOLVABLE $\in$ NP.

Proof. For necessity, let σ be a D-solution from C and define $x_{u,v}$ to be the number of pebbling steps in σ from u to v. Then, for each $u \in V$, the left hand side of (4.1) calculates the resulting number of pebbles at u — how many started at u, plus how many came to u, minus how many left u — which is at least $D(u)$ because σ solves D.

For sufficiency, we begin with a solution to (4.1) that has the smallest sum of all its variables, $\sum_{u \in V} \sum_{v \in V-\{u\}} x_{u,v}$, and construct from it a D-solution σ. Create the directed graph $\vec{G}$ by replacing every edge of G by two arcs in each direction, labeling each arc uv by the value $x_{u,v}$, which we think of as the number of pebbling steps of σ from u to v, and removing all arcs with label 0. Because of the minimum sum condition, $\vec{G}$ has no oriented cycles (see exercises). All that remains is to define σ by listing the pebbling steps in order so that each configuration along the way is nonnegative, which we also leave to the exercises. □

Now we will show that D-SOLVABLE (in particular, r-SOLVABLE) is NP-hard. Let $\mathcal{H}$ be a hypergraph with 2^{t+2} vertices $V(\mathcal{H})$ and edges $E(\mathcal{H}) = \{e_1, \ldots, e_k\}$. Define

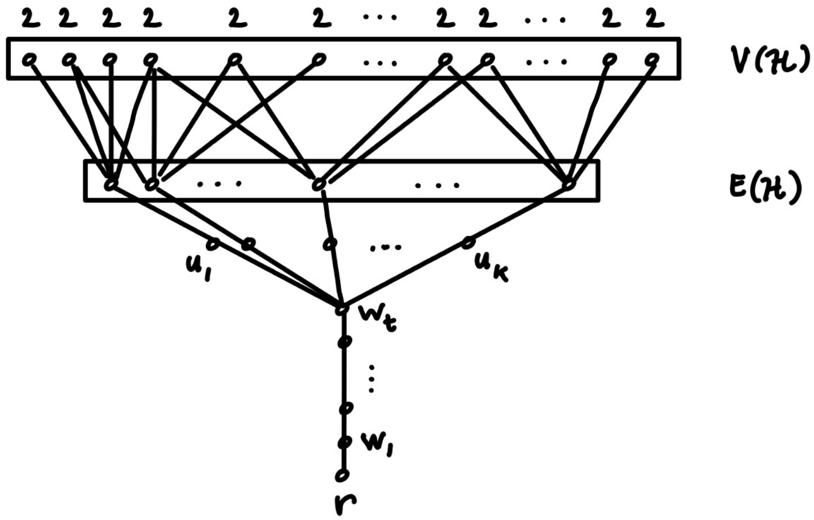

FIGURE 12. The pebbling graph $G_{\mathcal{H}}$ of a 4-uniform hypergraph $\mathcal{H}$ on 2^{t+2} vertices, with configuration $C_{\mathcal{H}}$ indicated.

the *pebbling graph* $G = G_{\mathcal{H}}$ as follows (See Figure 12). The vertices of G are given by $V(G) = V(\mathcal{H}) \cup E(\mathcal{H}) \cup \{u_1, \ldots, u_k\} \cup \{r, w_1, \ldots, w_t\}$. The edges of G include $ve_i \in E(G)$ for every $v \in e_i$, as well as the paths $w_t u_i e_i$ for every $i \leq k$ and the path $rw_1 \cdots w_t$.

Theorem 4.2. [56] *Let $\mathcal{H}$ be a 4-uniform hypergraph on 2^{t+2} vertices with pebbling graph $G = G_{\mathcal{H}}$. Define the configuration $C = C_{\mathcal{H}}$ on G by $C(v) = 2$ for all $v \in V(\mathcal{H})$ and $C(v) = 0$ otherwise. Then C is r-solvable if and only if $\mathcal{H}$ has a perfect matching. Hence r-*SOLVABLE *is* NP*-complete.*

Proof. The first thing to notice is that $wt(C) = 1$, which means that every r-solution is greedy. This means that there will never be more than 2 pebbles on any vertex in $V(\mathcal{H})$, and so there will never be more than 4 pebbles on any vertex e_i. Thus each e_i can contribute at most one pebble to w_t, and will contribute one if and only if it receives one pebble from each of the four vertices of $V(\mathcal{H})$ that it contains as an edge of $\mathcal{H}$. Since each vertex in $V(\mathcal{H})$ can contribute a pebble to at most one e_i, those e_i that do contribute a pebble to w_t form a matching in $\mathcal{H}$. Now, solving r requires that 2^t pebbles reach w_t, which happens if and only if the matching is perfect. □

Milans and Clark [59] also achieve this result via reduction from *canonical* 3SAT(which we write 3SATC), in which each formula Φ considered has at least 2 clauses, with either 2 or 3 variables per clause, and each variable appearing exactly once in its negative form and either once or twice in its positive form. They build a *pebbling graph* $G = G_{\Phi}$ with configuration $C = C_{\Phi}$ and root r (see Figure 13) so that C is r-solvable on G if and only if Φ is satisfiable. We define G by starting with the path $v_{x^1} v_{\overline{x}} v_{x^2}$ for each variable x in Φ, with pebbles $C(v_{x^1}, v_{\overline{x}}, v_{x^2}) = (2, 0, 2)$. The vertex v_{x^i} represents the i^{th} occurrence of x in Φ, while $v_{\overline{x}}$ represents the occurrence of $\overline{x}$ in Φ. For every OR (resp. AND) in Φ we add a representative vertex adjacent to each of the variables or clauses it disjoins (resp. conjoins) and place 1 (resp. 0) pebbles on it. After the final AND vertex has been placed we add the root vertex r adjacent to it.

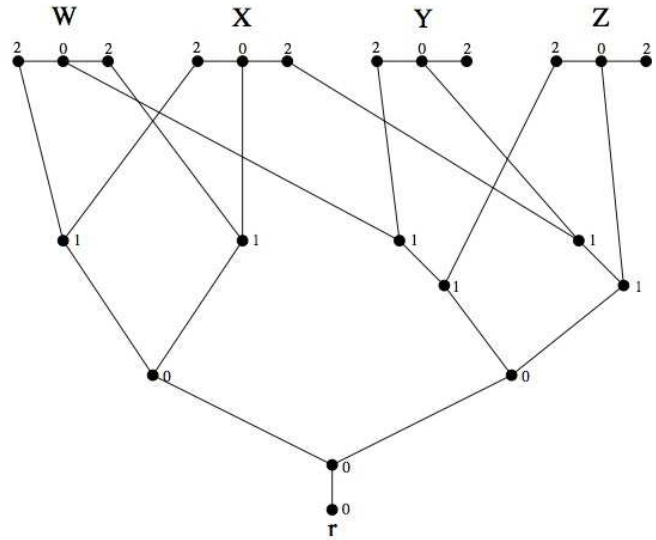

FIGURE 13. A pebbling graph G_Φ with configuration C_Φ for $\Phi = (w \vee x) \wedge (w \vee \overline{x}) \wedge (\overline{w} \vee y \vee z) \wedge (x \vee \overline{y} \vee \overline{z})$.

As with most NP-complete problems, one looks for classes of graphs over which the problem is polynomial. Of course, on trees r-SOLVABLE is linear: make greedy steps along the edges farthest from r until you either reach r or not. The hypergraph construction above shows that bipartite graphs are not such a class, even when $\max C = 2$ (and [59] further restrict the graphs to $\Delta = 3$). Diameter two graphs are a natural class to consider, but Cusack, et al., found that they are no easier to work with, reducing r-SOLVABLE in this instance from 3SATC as well.

Theorem 4.3. [21] *When restricted to the class of diameter two graphs, r-SOLVABLE remains NP-complete.*

Of course, if $|C| > \pi(G, r)$ then we know C is r-solvable, so it is the small configurations that pose the challenge. Bekmetjev and Cusack [4] show, however, that the situation improves if we again consider connectivity: the problem is polynomial over diameter two graphs of fixed connectivity. On the other hand, even if we know that a configuration solves r, it may still be difficult to find an r-solution. In this case, diameter two graphs are more friendly, even for k-fold solutions.

Theorem 4.4. [45] *If $\mathsf{diam}(G) = 2$, $|E(G)| = m$, and $|C| \geq \pi(G, r) + 4k - 4$ then a k-fold r-solution can be found from C in at most $6n + \min\{3k, m\}$ steps.*

A much more complicated reduction yields a negative result for planar graphs as well.

Theorem 4.5. [20] *When restricted to the class of planar graphs, r-SOLVABLE remains NP-complete.*

However, Cusack et al. show the following surprising positive result.

Theorem 4.6. [20] *When restricted to the class of diameter two planar graphs we have SOLVABLE $\in$ P.*

Their proof relies on a result of [4] that states that if a configuration C has b big vertices then the solvability of C can be determined in $O(b!n^{2b-1}m)$ time. For $b \leq 3$

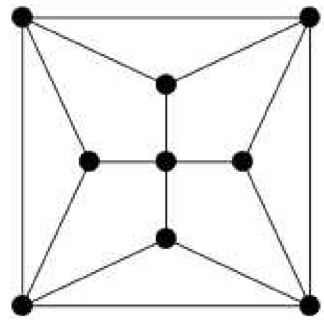

FIGURE 14. The unique diameter two graph with domination number 3.

they use this algorithm, and they show that if $b \geq 4$ then C is solvable. The proof of this statement is arranged according to the value of $\mathsf{dom}(G)$. A theorem of Goddard and Henning [38] states that $\mathsf{dom}(G) \leq 3$, with equality if and only if G is the graph in Figure 14. It is easy to see that placing two pebbles on each of any four vertices of this graph will solve any root. We leave it to the exercises that doing the same on a planar graph of domination number 1 or 2 will have the same effect. Only the $\mathsf{dom}(G) = 2$ case uses planarity, as a subdivision of $K_{3,3}$ comes into play.

4.2. **Pebbling and Optimal Pebbling Numbers.** We define **PEBBLINGNUMBER** to be the problem of calculating $\pi(G)$. As an example of the interesting phenomenon at play here, we can calculate the pebbling number of a diameter two graph quickly without being able to decide quickly if a configuration is solvable.

Theorem 4.7. [45] *Calculating $\pi(G)$ when G is a diameter two graph can be done in $O(n^4)$ time.*

The algorithm for this result relies on the pyramidal characterization of Theorem 2.9 and improves on the earlier $O(n^3m)$ algorithm of [4].

Next we define the related **UPPERBOUND** to be the problem of deciding, for given k and configuration D on a graph G, if $\pi(G, D) \leq k$. Denote by Π_2^{P} the class of decision problems computable in polynomial time by a coNP machine equipped with an NP-complete oracle.

Theorem 4.8. [59] UPPERBOUND *is Π_2^{P}-complete.*

We leave that UPPERBOUND $\in \Pi_2^{\mathsf{P}}$ to the exercises. To show that UPPERBOUND is Π_2^{P}-hard, Milans and Clark reduce from ∀∃3SATC. This is the natural class to consider because proving that $\pi(G) \leq k$ requires one to show for all r that every configuration of size at least k has an r-solution. As a consequence of this theorem, UPPERBOUND is both NP-hard and coNP-hard, which implies that it is in neither NP nor coNP unless NP=coNP.

Finally, we return to consider graph classes over which PEBBLINGNUMBER is polynomial. Alcón et al. [1] show that calculating $\pi(G, r)$ when G is a split graph can be done in linear time, based on the fact that recognizing whether or not G is r-Pereyra can be done in linear time. Indeed, G r-Pereyra requires r to be a degree 2 cone vertex. Now let X be the set of cut vertices of G, W be the set of degree 2 vertices of G whose neighbors are in $G - X$, and define the graph $H = H(G)$ to have vertices $\cup_{v \in W} N(v)$, where $N(v)$ is the set of neighbors of v in G, and edges $\{N(v)\}_{v \in W}$. Then

G is r-Pereyra if and only if H has a triangle including the edge $N(r)$. It is not hard to see that this whole process is linear (see exercises). Then one uses the following three lemmas to show that calculating $\pi(G,r)$ can be done in linear time. The lemmas are proved in [1] and are among those used to prove Theorem 2.17. The function $t(G,r)$ is complicated to write down briefly, but is easily calculated from knowing the degrees and number of cut vertices of G.

Lemma 4.9. *If G is a split graph and $r \in K$ then $\pi(G,r) = n + \mathrm{x}^r$, where x^r is the number of cut vertices different from r.*

Lemma 4.10. *If G is a split graph and r is a cone vertex with $\mathsf{ecc}(r) = 2$, then $\pi(G,r) = n + \mathrm{x} + \psi$, where x is the number of cut vertices and $\psi = \psi(G,r)$ is 1 if G is r-Pereyra and 0 otherwise.*

Lemma 4.11. *If G is a split graph and r is a cone vertex with $\mathsf{ecc}(r) = 3$, then $\pi(G,r) = t(G,r) + \phi(G,r)$, where $\phi(G,r) = 1$ if G is r-Phoenix and 0 otherwise.*

Now, calculating $\pi(G)$ and $\pi(G,r)$ are polynomially equivalent, but it may be possible to calculate $\pi(G)$ faster than by calculating $\pi(G,r)$ for every r. In fact, for split graphs the naïve algorithm would be quadratic, but the following result does better.

Theorem 4.12. [1] *Calculating $\pi(G)$ when G is a split graph can be done in $O(n^\beta)$ time, where $\omega \cong 2.37$ is the exponent of matrix multiplication and $\beta = 2\omega/(\omega+1) \cong 1.41$.*

Obviously the proof uses Theorem 2.17. The crucial time saving piece in the proof of Theorem 4.12 comes from showing that recognizing if G is Pereyra can be done in $O(n^\beta)$ time, as G is Pereyra if and only if $H(G)$ contains a triangle, which can be decided in that time by using an algorithm of Alon, Yuster, and Zwick. The only other nontrivial part comes when $\mathrm{x} = 0$, in which case there is a linear algorithm that either finds a degree 2 cone vertex r having some degree at most 3 vertex at distance 3 from it or concludes that none exist. We leave this algorithm to the exercises.

Finally we define **OPTIMALPEBBLINGNUMBER** to be the problem of deciding if $\pi^*(G) \le k$, and we present, without proof, the one result in this area.

Theorem 4.13. [59] *The problem* OPTIMALPEBBLINGNUMBER *is* NP*-complete.*

Exercises.

(1) Prove that SOLVABLE is linear on trees.
(2) Convert Chung's proof of $\pi(Q^d) = n$ to a linear algorithm that solves configurations of size at least n.
(3) Show that the digraph $\vec{G}$ in the proof of Theorem 4.1 is acyclic.
(4) Finish the proof of Theorem 4.1 by ordering the steps of σ.
(5) Prove that a 3SATC formula Φ is satisfiable if and only if C_Φ is r-solvable on G_Φ.
(6) Let G be a diameter two planar graph with $\mathsf{dom}(G) \le 2$. Suppose C has at least 4 vertices with at least 2 pebbles each. Show that C is solvable.
(7) Prove that UPPERBOUND $\in \Pi_2^{\mathsf{P}}$.
(8) Devise a linear algorithm to construct a maximum path partition of a tree.
(9) Show that the algorithm outlined for recognizing if G is r-Pereyra is linear.
(10) Prove the lower bound $\pi(G,r) \ge n + \mathrm{x}^r$ from Lemma 4.9.
(11) Use the Weight Function Lemma 1.8 to prove the upper bound $\pi(G,r) \le n + \mathrm{x}^r$ from Lemma 4.9.

(12) Prove the lower bound $\pi(G,r) \geq n + \mathrm{x} + \psi$ from Lemma 4.10.
(13) Show that calculating $\pi(G,r)$ can be done in linear time.
(14) Devise an linear algorithm that handles the case $\mathrm{x} = 0$ in the proof of Theorem 4.12.
(15) Show that $\mathsf{OptimalPebblingNumber} \in \mathsf{NP}$.
(16) (Open) Is $\mathsf{PebblingNumber} \in \mathsf{P}$ when restricted to interval graphs of fixed diameter?
(17) (Open) Is r-$\mathsf{Solvable} \in \mathsf{P}$ when restricted to the class of cubes?
(18) (Open) Is r-$\mathsf{Solvable} \in \mathsf{P}$ when restricted to the class of split graphs?
(19) (Open) Is it possible to decide in polynomial time if a fixed diameter chordal graph is Class 0?
(20) (Open) Find classes of graphs over which $\mathsf{OptimalPebblingNumber} \in \mathsf{P}$.

5. Thresholds

The probabilistic model of pebbling studies the typical case; that is, small configurations are usually unsolvable and large configurations are usually solvable — at roughly how many pebbles is the transition? In other words, instead of randomizing the graph, as in Theorem 2.14, we randomize the configuration.

We assume that all sequences $\mathcal{G} = (G_1, \ldots, G_k, \ldots) = (G_k)_{k\geq 1}$ of graphs considered have an increasing number of vertices $n = n_k = n(G_k)$. The sequences of complete graphs, stars, paths, cycles, and cubes are denoted $\mathcal{K}$, $\mathcal{S}$, $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{Q}$, respectively. The sequence of graph products is written $\mathcal{G}\square\mathcal{H} = (G_k \square H_k)_{k\geq 1}$, with $\mathcal{G}^2 = \mathcal{G}\square\mathcal{G}$. For sets of functions A and B on the integers we write $A \lesssim B$ to mean that $a \in O(b)$ for every $a \in A, b \in B$.

Let $C_k : [n] \to \mathbb{N}$ denote a configuration on $V(G_k)$ and, for a function $h : \mathbb{N} \to \mathbb{N}$ and fixed $n = n_k$, define the uniform probability space $X_{n,h}$ of all configurations C_k of size $h = h(n)$. Define the probability $\mathbf{P}_n^+ = \mathbf{Pr}[C_k \text{ is solvable on } G_k]$ and let $t : \mathbb{N} \to \mathbb{N}$ be any function. We say that t is a *pebbling threshold* for $\mathcal{G}$, and write $\tau(\mathcal{G}) = \Theta(t)$, if $\mathbf{P}_n^+ \to 0$ whenever $h(n) \ll t(n)$ and $\mathbf{P}_n^+ \to 1$ whenever $h(n) \gg t(n)$. For example, the solvability of small configurations on K_k is equivalent to the labelled version of Feller's Birthday Problem (as well as to hashing collisions in computer science); that is, if $|C_k| < k$ then C_k is solvable if and only if some vertex is big. Thus $\tau(\mathcal{K}) = \Theta(\sqrt{n})$.

For notational convenience we will assume that functions like logarithms and square roots take on their nearest integer values.

5.1. Existence.
A priori there is no reason that $\mathbf{P}_n^+$ must tend to either 0 or 1 for every h. In other contexts, probabilities of statements holding true can tend to other constants. For example, Winkler [77] proved that, in a random 2-dimensional poset, the probability that some pair of maximal and minimal elements are incomparable tends to 3/4. However, Bekmetjev et al. show that the 0-1 law does hold for graph pebbling.

Theorem 5.1. [3] *Every graph sequence $\mathcal{G}$ has nonempty threshold $\tau(\mathcal{G})$.*

For the moment we will assume this so that we don't complicate the statements of some preliminary results. As in the Subgraph Bound, because the r-solvability of a configuration is not destroyed by adding edges, we have the following threshold result.

Lemma 5.2. [22] **[Subgraph Lemma]** *If H_k is a spanning subgraph of G_k for all k then $\tau(\mathcal{G}) \lesssim \tau(\mathcal{H})$.*

This is what places every threshold in the range $\tau(\mathcal{K}) \lesssim \tau(\mathcal{G}) \lesssim \tau(\mathcal{P})$ (see exercises). For some time it was wondered if thresholds were monotone with respect to pebbling numbers; that is, if $\pi(G_k) \le \pi(H_k)$ for all k implied $\tau(cG) \lesssim \tau(\mathcal{H})$. But counterexamples of Björklund and Holmgren [7] disproved this (see exercises).

We now sketch a proof of Theorem 5.1. In many ways the proof mimics that of the result of Bollobás and Thomason [10] that monotone graph properties (such as containing triangles or having no hamilton cycle) have random graph thresholds — their result actually holds more generally for monotone families of subsets — and in other ways a new approach was needed.

Proof of Theorem 5.1. Note that a configuration of size t is simply a t-multiset of n vertices, the number of which we denote by $\left\langle {n \atop t} \right\rangle = \binom{n+t-1}{t}$. We write $\mathscr{M}_n$ for the set of all multisets of $[n]$ and $\mathscr{M}_n(t)$ for those of size t. A family $\mathscr{F}_n \subseteq \mathscr{M}_n$ (with $\mathscr{F}_n(t) = \mathscr{F}_n \cap \mathscr{M}_n(t)$) is *increasing* (resp. *decreasing*) if $E \supseteq F \in \mathscr{F}_n$ (resp. $E \subseteq F \in \mathscr{F}_n$) implies $E \in \mathscr{F}_n$. Notice that the family of all solvable configurations of a graph is increasing. A randomly chosen element of $\mathcal{M}_n(t)$ has probability $\mathbf{P}_t(\mathscr{F}_n) = |\mathscr{F}_n(t)|/\left\langle {n \atop t} \right\rangle$ of being in $\mathscr{F}_n$. We define $t^* = t^*(n) = \min\{h \mid \mathbf{P}_h(\mathscr{F}_n) \ge 1/2\}$ and show that $t^* \in \tau(\mathscr{F})$, where $\mathscr{F} = (\mathscr{F}_n)_{n\ge 1}$.

A crucial tool is the following result of Clements and Lindström, extending an earlier result of Macauley, which is the multiset analog of of the celebrated Kruskal-Katona Theorem for the subset lattice. Given any subfamily $\mathscr{A} \subseteq \mathscr{M}_n(t)$, we define its *shadow* $\partial\mathscr{A} = \{B \in \mathcal{M}_n(t-1) \mid B \subseteq A \text{ for some } A \in \mathscr{A}\}$, with the repeated shadow operator defined by $\partial^{k+1} = \partial\partial^k$. For a multiset $A \in \mathscr{M}_n(t)$ and $i \in [n]$ let $A(i)$ denote the multiplicity of i in A. The *colexicographic (colex) order* on $\mathscr{M}_n(t)$ is defined by setting $A < B$ if $A \ne B$ and, for some $i \in [n]$, $A(i) < B(i)$ while $A(j) = B(j)$ for $j > i$.

Proposition 5.3. [17] *If $\mathscr{F} \subseteq \mathscr{M}_n(t)$ and $\mathscr{G}$ consists of the first $|\mathscr{F}|$ elements of $\mathscr{M}_n(t)$ in the colex order then $|\partial^k\mathscr{F}| \ge |\partial^k\mathscr{G}|$ for all $k \ge 1$.*

In 1979 Lovász proved a continuous version of the Kruskal-Katona Theorem which was used by Bollobás and Thomason to prove their result. The next big tool is the following analogous continuous version of Clements-Lindström. For a nonnegative real number x let $\left\langle {x \atop t} \right\rangle = (x)(x-1)\cdots(x-t+1)/t!$.

Proposition 5.4. [3] *Suppose that $\mathscr{A} \in \mathscr{M}_n(t)$ and define x by $|\mathscr{A}| = \left\langle {x \atop t} \right\rangle$. Then $|\partial\mathscr{A}| \ge \left\langle {x \atop t-1} \right\rangle$.*

Here is where the proof detours slightly from the Bollobás-Thomason approach. Compared to the levels of the subset lattice, $|\mathscr{M}_n(t)|$ grows too fast for the relevant probabilities to tend to 0 quickly enough with these shadow estimates. So instead two special reference families are needed: (1) $\mathscr{M}_n^b(t) = \{A \in \mathscr{M}_n(t) \mid A(n) < b\}$ $(1 \le b \le n)$, and (2) $\mathscr{N}_n^b(t) = \{A \in \mathscr{M}_n(t) \mid A(n-b+1) = \cdots = A(n) = 0\}$ $(1 \le b \le n-1)$. Note that each of these families is some initial segment of colex on $\mathscr{M}_n(t)$. Also, $\partial^k\mathscr{M}_n^b(t) = \mathscr{M}_n^b(t-k)$ and $\partial^k\mathscr{N}_n^b(t) = \mathscr{N}_n^b(t-k)$ for all $k \ge 1$.

We start with an increasing family $\mathscr{F}_n$ and define $\overline{\mathscr{F}_n} = \mathscr{M}_n - \mathscr{F}_n$. The idea is to compare $\overline{\mathscr{F}_n}(t)$ with the appropriate reference family at or near level t^* in order to show that $\mathbf{P}_t(\overline{\mathscr{F}_n})$ tends to 0 or 1 as needed. We use the estimates in Exercises 6 and 7 in the following four cases and leave the details to the exercises. Take any $\omega = \omega(n)\to\infty$.

(1) Let $t = t^*/\omega$ and show that $\mathbf{P}_t(\overline{\mathscr{F}_n})\to 1$.

(a) If $t^* \geq 2n-1$ then set $b = t^*/(2n-1)$. Then $\mathbf{P}_{t^*-1}(\mathscr{M}_n^b) < \mathbf{P}_{t^*-1}(\overline{\mathscr{F}_n})$ and $\mathbf{P}_t(\overline{\mathscr{F}_n}) \geq 1 - (t/(n+t-1))^b$. If $n-1 \geq t^*/\sqrt{\omega}$ this is at least $1 - 1/(\sqrt{\omega}+1)$; otherwise it is at least $1 - e^{-\sqrt{\omega}/10}$.

(b) If $t^* \leq 2n-2$ then set $b = (n+t^*-2)/(t^*-1)$. Then $\mathbf{P}_{t^*-1}(\mathscr{N}_n^b) < \mathbf{P}_{t^*-1}(\overline{\mathscr{F}_n})$ and $\mathbf{P}_t(\overline{\mathscr{F}_n}) \geq e^{-tb/\omega(n-b)} \geq e^{-6/\omega}$.

(2) Now let $t = \omega t^*$ and show that $\mathbf{P}_t(\overline{\mathscr{F}_n}) \to 0$.

(a) If $t^* \geq n/2$ then set $b = (n+t-1)/(n-1)$. Then $\mathbf{P}_{t^*}(\mathscr{M}_n^b) \geq \mathbf{P}_{t^*}(\overline{\mathscr{F}_n})$ and $\mathbf{P}_t(\overline{\mathscr{F}_n}) \leq 1 - e^{-8/\omega}$.

(b) If $t^* \leq (n-1)/2$ then set $b = n/(2t^*+1)$. Then $\mathbf{P}_{t^*}(\mathscr{N}_n^b) \geq \mathbf{P}_{t^*}(\overline{\mathscr{F}_n})$ and $\mathbf{P}_t(\overline{\mathscr{F}_n}) \leq ((n-1)/(n+t-1))^b$. If $n-1 \leq t^*/\sqrt{\omega}$ this is at most $1/(1+\sqrt{\omega})$; otherwise it is at least $e^{-\sqrt{\omega}/12}$.

□

5.2. **Calculations.** The most basic techniques for calculation bounds for thresholds involve packing and covering the vertices of a graph by its subgraphs, most often by balls: fixed radius neighborhoods of vertices. An upper bound comes from showing that, for most large configurations, every ball will have enough pebbles in it to be solved locally. A lower bound comes from showing that, for most small configurations, some ball will be empty and (by using weight functions) not enough pebbles can amass at its boundary to reach its center. It helps to be able to cap the number of pebbles at each vertex in order for the argument to be successful. Usually different radius balls are required for these arguments. We begin with the following general upper bound.

Proposition 5.5. [22] *For all $\epsilon > 0$, every graph sequence $\mathcal{G}$ satisfies $\tau(\mathcal{G}) \subseteq O(n^{1+\epsilon})$.*

Proof. We use the fact that any graph H on l vertices with spanning tree T has $\pi(H) \leq \pi(T) \leq \pi(P_l) < 2^l$. Let $c, \delta > 0$, $t = cn^{1+\epsilon}$, $l = (1+\delta)/\epsilon$, and $s = 2^l$. Let $G = G_k$ be a graph on $n = n_k$ vertices with randomly chosen configuration $C = C_k$ of size t. For each vertex v choose a connected l-vertex subgraph $H(v)$ of G containing v. Denote by E_v the event that $C(H(v)) < s$. The result will follow from showing that $\mathbf{Pr}[\exists v\ E_v] \to 0$.

$$\begin{aligned}
\mathbf{Pr}[\exists v\ E_v] &\leq n \sum_{i=0}^{s-1} \mathbf{Pr}[C(H(v)) = i] \\
&= n \sum_{i=0}^{s-1} \left\langle {l \atop i} \right\rangle \left\langle {n-l \atop t-i} \right\rangle \Big/ \left\langle {n \atop t} \right\rangle \\
&\leq \frac{n}{\left\langle {n \atop t} \right\rangle} \sum_{i=0}^{s-1} \left\langle {l \atop i} \right\rangle \left\langle {n-l \atop t} \right\rangle \left(\frac{t}{n-l+t-1} \right)^i \\
&\leq n \left(\frac{n}{t} \right)^l \sum_{i=0}^{s-1} \left\langle {l \atop i} \right\rangle \left(\frac{t}{n-l+t-1} \right)^i \\
&\leq n(c'n^{-\epsilon l}) \sum_{i=0}^{s-1} \left\langle {l \atop i} \right\rangle \ \leq\ c''n^{-\delta} \to 0.
\end{aligned}$$

□

For tighter results we will need the following elementary inequalities.

Markov's Inequality. *If X is any nonnegative random variable and $a > 0$, then* $\mathbf{Pr}[X \geq a] \leq \mathbf{E}[X]/a$.

The use of Markov's Inequality is often referred to as the *first moment method*. For many of our purposes X will take on integer values, so that $\mathbf{Pr}[X > 0] = \mathbf{Pr}[X \geq 1] \leq \mathbf{E}[X]$. From this it follows that if $\mathbf{E}[X]\to 0$ then $\mathbf{Pr}[X = 0]\to 1$.

Chebyshev's Inequality. *If X is any nonnegative random variable and $a > 0$, then* $\mathbf{Pr}[|X - \mathbf{E}[X]| \geq a] \leq \mathbf{Var}[X]/a^2$.

The use of Chebyshev's Inequality is often referred to as the *second moment method*. Typically it is used to show that $\mathbf{Pr}[X = 0] \leq \mathbf{Pr}[|X - \mathbf{E}[X]| \geq \mathbf{E}[X]] \leq \mathbf{Var}[X]/\mathbf{E}[X]^2$. From this it follows that if $\mathbf{E}[X]\to\infty$ and $\mathbf{Var}[X] \in o(\mathbf{E}[X]^2)$ then $\mathbf{Pr}[X > 0]\to 1$.

A relatively simple lower bound for paths is given by the following.

Theorem 5.6. [22] *The threshold for the sequence of paths satisfies $\tau(\mathcal{P}) \subseteq \Omega(n)$.*

Proof. Let $t = n/\omega$ for some $\omega\to\infty$ and let $C = C_n$ be a random configuration of size t on P_n. We know from Theorem 1.7 that C is r-solvable for the leaf r if and only if $wt(C) \geq 1$. Now, every vertex v has $\mathbf{E}[C(v)] = t/n = 1/\omega\to 0$, so $\mathbf{E}[wt(C)] = \mathbf{E}[C(v)]\sum_i 2^{-i} < 2/\omega\to 0$. Thus $\mathbf{Pr}[wt(C) \geq 1]\to 0$ by Markov's Inequality, completing the proof. □

In a series of papers [22, 3, 37, 25], efforts were made to pin down $\tau(\mathcal{P})$, the results of which are given below. Because the constant c is in the exponent, there is still a wide gap between the lower and upper bounds.

Theorem 5.7. [37, 25] *For every constant $c > 1$, we have* $\tau(\mathcal{P}) \subseteq \Omega\left(n2^{\sqrt{\lg n}/c}\right) \cap O\left(n2^{c\sqrt{\lg n}}\right)$.

Obtaining the upper bound with $c = 2$ was accomplished in [3] by using the same technique as that which proved Proposition 5.5, by partitioning P_n into paths of length $\sqrt{\lg n}$ (see exercises). Dropping c near 1 requires much more finesse. Obtaining the lower bound with any $c > \sqrt{2}$ was also done in [3], with an argument we now present.

Let $u = \sqrt{\lg n}/c$, $t = n2^u$, and $p = (1 + \epsilon)2^u \lg n$ for some $\epsilon > 0$. Let C be a random configuration of size t and denote by F_v be the event that $C(v) > p$. The following claim caps $\max C$; we leave its proof as an exercise.

Claim 5.1. With u, t, ϵ, and p as above we have $\mathbf{Pr}[\exists v\ F_v]\to 0$.

Now partition P_n into q paths Q_i of length $l = \sqrt{2\lg n}$, and denote by E_i the event that Q_i is empty.

Claim 5.2. With u, t, q, and l as above we have $\mathbf{Pr}[\exists i\ E_i]\to 1$.

Proof. Write $s = 2^u$ so that $t = ns$ and note that $n \gg ls^l e^{l/s}$. Let X_i be the indicator variable of E_i ($X_i = 1$ if Q_i is empty, and 0 otherwise), and set $X = \sum_{i=1}^{q} X_i$. Then

$\mathbf{E}[X] = q\mathbf{E}[X_i] = q\mathbf{Pr}[C(Q_i)=0] = \binom{n}{l}\left\langle{n-l \atop t}\right\rangle / \left\langle{n \atop t}\right\rangle$, so

$$\begin{aligned}\mathbf{E}[X] &\geq \binom{n}{l}\left(\frac{n-l}{t+n-l}\right)^{l}\\ &= \binom{n}{l}\left(\frac{t}{n-l}+1\right)^{-l}\\ &\sim \left(\frac{n}{ls^{l}}\right)(1+l/n+1/s)^{-l}\\ &\sim \frac{ne^{-l^2/n}}{ls^{l}e^{l/s}} \;\to\infty,\end{aligned}$$

and the second moment method applies. Now we compute

$$\begin{aligned}\mathbf{Var}[X] &= \mathbf{E}[X^2]-\mathbf{E}[X]^2\\ &= \sum_{i,j}\mathbf{E}[X_iX_j]-\sum_{i,j}\mathbf{E}[X_i]\mathbf{E}[X_j]\\ &\leq \sum_{i}\mathbf{E}[X_i^2]\\ &= \sum_{i}\mathbf{E}[X_i]\\ &= \mathbf{E}[X] \;\ll\; \mathbf{E}[X]^2.\end{aligned}$$

We leave explaining the middle inequality to the exercises. ◇

Let Q be an empty path almost guaranteed by Claim 5.2. Because Claim 5.1 almost guarantees that each vertex has at most p pebbles, the weight of C to the left of the left endpoint of Q is at most $p\sum_i 2^i < p < 2^{l/2}$ for large enough n, which is not enough to reach the center of Q. The same can be said for the pebbles to the right of Q, which finishes the proof for the case that $c > \sqrt{2}$.

When c is closer to 1 we cannot expect to find such large empty subpaths so, in their proof of the lower bound in Theorem 5.7, Czygrinow and Hurlbert replace emptiness by a bowl-shaped configuration that has an empty middle and increasing blocks as the distance from the middle increases. The rate of increase in relation to the size of the blocks is controlled in such a way that one can still use weight arguments to ensure that such bowl-shaped configurations (and the pebbles outside of them) cannot solve their center vertex. Then they prove that configurations bounded by such bowls almost surely exist.

As complete graphs and paths are at the opposite ends of the threshold spectrum, it seems reasonable to try to combine the two in order to build graph sequences of any threshold in between. Czygrinow and Hurlbert [25] achieved this up to $\Theta(n)$, but the region above that remains open (see exercises). For $k \geq 1$ and $l = l(n)$ define the *k-legged l-spider* $S^k_{n,l}$ by K_{n-kl} with an endpoint of each of k copies of P_l adjacent to k distinct vertices of the clique, and let $\mathcal{S}^k_l = (S^k_{n,l})_{n\geq 1}$. They prove that the sequence $\mathcal{S}^1_l$ with the appropriate value of l yields the following result.

Theorem 5.8. [25] *Let t_1 and t_2 be functions satisfying $\tau(\mathcal{K}) \lesssim t_1 \ll t_2 \lesssim \Theta(n)$. Then there is some graph sequence $\mathcal{G}$ such that $t_1 \lesssim \tau(\mathcal{G}) \lesssim t_2$.*

One line of research that parallels Class 0 questions is the search for graph sequences that have minimum threshold. An example is any sequence of graphs of high minimum degree.

Theorem 5.9. [23] *For $\delta = \delta(n)$ let $\mathcal{G}_\delta$ denote any sequence of graphs having minimum degree δ. If $n^{1/2} \ll \delta = \delta(n) \le n-1$ then $\tau(\mathcal{G}_\delta) \subseteq O(n^{3/2}/\delta)$. In particular, if in addition $\delta \in \Omega(n)$ then $\tau(\mathcal{G}_\delta) = \Theta(n^{1/2})$.*

The proof first constructs a partition of the vertices into $O(n/\delta)$ stars of size at least $\delta + 1$, along with an extra set of vertices that are each adjacent to some star. Second, they use the second moment method to prove that every star has at least 8 big vertices, from which we can place at least 4 pebbles on every star center, and then solve any vertex because the star centers form a distance 2 dominating set.

Another example is the sequence of squares of complete graphs.

Theorem 5.10. [5] *The sequence of squares of cliques has threshold $\tau(\mathcal{K}^2) = \Theta(\sqrt{n})$.*

This example leads to a second line of research that parallels Graham's Conjecture 2.2. In order to state an analogous conjecture for thresholds one needs to rescale threshold functions of products of graph sequences in terms of the new number of vertices $n(G_k \square H_k) = n(G_k) \square n(H_k)$; for example, in Fact 5.10 we have $\sqrt{n^2} = \sqrt{n}^2$. Products of paths are another instance of evidence for making such a conjecture.

Theorem 5.11. [24] *Suppose that $t \in \tau(\mathcal{P})$ and $t_2 \in \tau(\mathcal{P}^2)$. Then $t_2(n) \in O\left(t\left(\sqrt{n}\right)^2\right)$.*

They actually show the more general result that if $\mathcal{P}^d = (P_n^d)_n$ and $t_d \in \tau(\mathcal{P}^d)$ then $t_d(n) \in O(\tau(n^{1/d})^d)$. Evidence for a potential counterexample to such a product conjecture may come from the sequence of cubes, for which only the bounds below are known.

Theorem 5.12. [2, 27] *For all $\epsilon > 0$ the sequence of cubes has threshold $\tau(\mathcal{Q}) \subseteq \Omega(n^{1-\epsilon}) \cap O(n/(\lg\lg n)^{1-\epsilon})$.*

We compare these results by defining, for $l = l(n)$ and $d = d(n)$, the graph sequence $\mathcal{P}_l^d = (P_{l(n)}^{d(n)})_{n\ge 1}$, where $P_l^d = (P_l)^d$. Note that $\mathcal{P}_n^d = \mathcal{P}^d$, $\mathcal{P}_2^n = \mathcal{Q}$ (which we write $\mathcal{P}_2$), and $\tau(\mathcal{P}_2) \ll n \ll \tau(\mathcal{P}^d)$ for fixed d. In [24] we find the reasonable conjecture that $\tau(\mathcal{P}_l) \in o(n)$ for fixed l.

Exercises.

(1) Prove that $\tau(\mathcal{K}) = \Theta(\sqrt{n})$.
(2) Let $\mathcal{S} = \{S_n\}$ be the sequence of stars. Prove that $\tau(\mathcal{S}) = \Theta(\sqrt{n})$.
(3) Prove that any sequence $\mathcal{T}$ of trees has $\tau(\mathcal{T}) \lesssim \tau(\mathcal{P})$.
(4) Prove that any sequence $\mathcal{G}$ of Class 0 graphs has $\tau(\mathcal{G}) \subseteq O(n)$.
(5) Prove that any sequence $\mathcal{G}_d$ of diameter d graphs has $\tau(\mathcal{G}_d) \subseteq O(n)$.
(6) Prove for all $1 \le b \le t$ that $1 - e^{-b(n-1)/(n+t-1)} \le \mathbf{P}_t(\mathscr{M}_n(t,b)) \le 1 - e^{-b(n-1)/(t-b+1)}$.
(7) Prove for all $1 \le b \le n-1$ that $e^{-tb/(n-b)} \le \mathbf{P}_t(\mathscr{N}_n(t,b)) \le e^{-tb/(n+t-1)}$.
(8) Complete the proof of Theorem 5.1 by filling in the details of the calculations in the final four cases.
(9) Prove Claim 5.1.
(10) Explain the middle inequality of the variance calculation in the proof of Claim 5.2.
(11) Let $\epsilon = \epsilon(n) > 1/2$ be any function such that $n^\epsilon \ll n$, and set $l = (2\epsilon - 1)\lg n$. Prove that $\tau(\mathcal{S}_l^1) = \Theta(n^\epsilon)$.
(12) Find the appropriate value of l for $\mathcal{G} = \mathcal{S}_l^1$ that yields Theorem 5.8.

(13) [7] Let $a = \lg n$ and $b = \frac{1}{2}\lg n + 1000$ and prove that $\pi(S^1_{n,a}) < \pi(S^2_{n,b})$ and $\tau(\mathcal{S}^1_a) < \tau(\mathcal{S}^2_b)$.
(14) Prove that $\tau(\mathcal{P}) \subseteq O\left(n2^{2\sqrt{\lg n}}\right)$.
(15) Prove Theorem 5.9.
(16) (Open) Determine $\tau(\mathcal{P})$.
(17) (Open) Determine $\tau(\mathcal{Q})$.
(18) (Open) Is it possible to extend Fact 5.8 to the range $\Omega(n) \cap \tau(\mathcal{P})$?
(19) (Open) Suppose that $\mathcal{G}$ is any graph sequence, $t \in \tau(\mathcal{G})$ and $t' \in \tau(\mathcal{G}^2)$. Is it true that $t'(n) \in O\left(t\left(\sqrt{n}\right)^2\right)$?
(20) (Open) Is it true that $\tau(\mathcal{P}_l) \subseteq o(n)$ for fixed l?
(21) (Open) Is there some $d = d(n) \rightarrow \infty$ such that $\tau(\mathcal{P}^d) = \Theta(n)$?

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